Chapter 4

Introduction to Compressible Flow

4.1 INTRODUCTION

In earlier chapters we developed the fundamental relations that are needed for the analysis of fluid flow. We have seen the special form that some of these take for the case of constant-density fluids. Our *main* interest now is in compressible fluids or gases. We shall soon learn that it is not uncommon to encounter gases that are traveling faster than the speed of sound. Furthermore, when in this situation, their behavior is quite different than when traveling slower than the speed of sound. Thus we begin by developing an expression for sonic velocity through an arbitrary medium. This relation is then simplified for the case of perfect gases. We then examine subsonic and supersonic flows to gain some insight as to why their behavior is different.

The Mach number is introduced as a key parameter and we find that for the case of a perfect gas it is very simple to express our basic equations and many supplementary relations in terms of this new parameter. The chapter closes with a discussion of the significance of h–s and T–s diagrams and their importance in visualizing flow problems.

4.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. Explain how sound is propagated through any medium (solid, liquid, or gas).
2. Define *sonic velocity*. State the basic differences between a *shock wave* and a *sound wave*.
3. (*Optional*) Starting with the continuity and momentum equations for steady, one-dimensional flow, utilize a control volume analysis to derive the general expression for the velocity of an infinitesimal pressure disturbance in an arbitrary medium.

4. State the relations for:
   a. Speed of sound in an arbitrary medium
   b. Speed of sound in a perfect gas
   c. Mach number
5. Discuss the propagation of signal waves from a moving body in a fluid by explaining *zone of action*, *zone of silence*, *Mach cone*, and *Mach angle*. Compare subsonic and supersonic flow in these respects.
6. Write an equation for the stagnation enthalpy (h_t) of a perfect gas in terms of enthalpy (h), Mach number (M), and ratio of specific heats (γ).
7. Write an equation for the stagnation temperature (T_t) of a perfect gas in terms of temperature (T), Mach number (M), and ratio of specific heats (γ).
8. Write an equation for the stagnation pressure (p_t) of a perfect gas in terms of pressure (p), Mach number (M), and ratio of specific heats (γ).
9. (*Optional*) Demonstrate manipulative skills by developing simple relations in terms of Mach number for a perfect gas, such as

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

10. Demonstrate the ability to utilize the concepts above in typical flow problems.

4.3 SONIC VELOCITY AND MACH NUMBER

We now examine the means by which disturbances pass through an elastic medium. A disturbance at a given point creates a region of compressed molecules that is passed along to its neighboring molecules and in so doing creates a *traveling wave*. Waves come in various *strengths*, which are measured by the amplitude of the disturbance. The speed at which this disturbance is propagated through the medium is called the *wave speed*. This speed not only depends on the type of medium and its thermodynamic state but is also a function of the strength of the wave. The *stronger* the wave is, the faster it moves.

If we are dealing with waves of *large amplitude*, which involve relatively large changes in pressure and density, we call these *shock waves*. These will be studied in detail in Chapter 6. If, on the other hand, we observe waves of *very small amplitude*, their speed is characteristic *only* of the medium and its state. These waves are of vital importance to us since sound waves fall into this category. Furthermore, the presence of an object in a medium can only be felt by the object's sending out or reflecting infinitesimal waves which propagate at the characteristic *sonic velocity*.

Let us hypothesize how we might form an infinitesimal pressure wave and then apply the fundamental concepts to determine the wave velocity. Consider a long constant-area tube filled with fluid and having a piston at one end, as shown in Figure 4.1. The fluid is initially at rest. At a certain instant the piston is given an

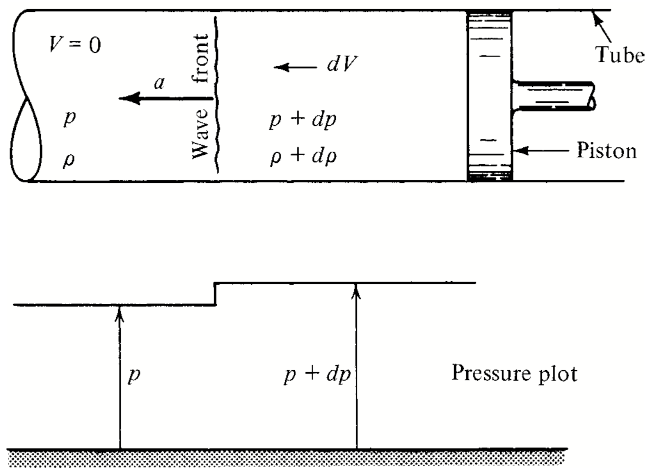

Figure 4.1 Initiation of infinitesimal pressure pulse.

incremental velocity dV to the left. The fluid particles immediately next to the piston are compressed a very small amount as they acquire the velocity of the piston.

As the piston (and these compressed particles) continue to move, the next group of fluid particles is compressed and the *wave front* is observed to propagate through the fluid at the characteristic *sonic velocity* of magnitude a. All particles between the wave front and the piston are moving with velocity dV to the left and have been compressed from ρ to $\rho + d\rho$ and have increased their pressure from p to $p + dp$.

We next recognize that this is a difficult situation to analyze. Why? Because it is *unsteady* flow! [As you observe any given point in the tube, the properties change with time (e.g., pressure changes from p to $p + dp$ as the wave front passes).] This difficulty can easily be solved by superimposing on the entire flow field a constant velocity to the right of magnitude a. *This procedure changes the frame of reference to the wave front as it now appears as a stationary wave.* An alternative way of achieving this result is to jump on the wave front. Figure 4.2 shows the problem that we now

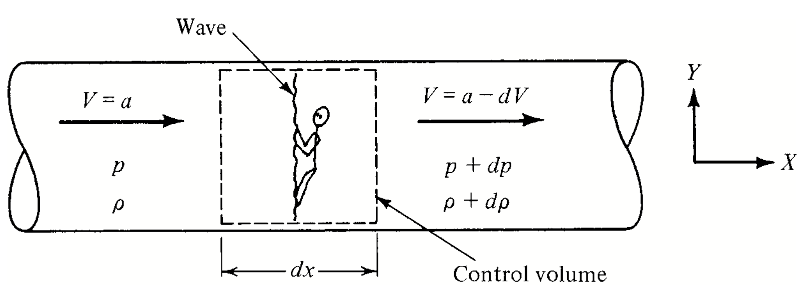

Figure 4.2 Steady-flow picture corresponding to Figure 4.1.

have. *Note that changing the reference frame in this manner does not in any way alter the actual (static) thermodynamic properties of the fluid, although it will affect the stagnation conditions*. Since the wave front is extremely thin, we can use a control volume of infinitesimal thickness.

Continuity

For steady one-dimensional flow, we have

$$\dot{m} = \rho A V = \text{const} \tag{2.30}$$

But $A = \text{const}$; thus

$$\rho V = \text{const} \tag{4.1}$$

Application of this to our problem yields

$$\rho a = (\rho + d\rho)(a - dV)$$

Expanding gives us

$$\cancel{\rho a} = \cancel{\rho a} - \rho\, dV + a\, d\rho - \underset{\nearrow \text{HOT}}{\cancel{d\rho\, dV}}$$

Neglecting the higher-order term and solving for dV, we have

$$dV = \frac{a\, d\rho}{\rho} \tag{4.2}$$

Momentum

Since the control volume has infinitesimal thickness, we can neglect any shear stresses along the walls. We shall write the x-component of the momentum equation, taking forces and velocity as positive if to the right. For steady one-dimensional flow we may write

$$\sum F_x = \frac{\dot{m}}{g_c}(V_{\text{out}_x} - V_{\text{in}_x}) \tag{3.46}$$

$$\cancel{p}A - (\cancel{p} + dp)A = \frac{\rho A a}{g_c}\,[(\cancel{a} - dV) - \cancel{a}]$$

$$A\, dp = \frac{\rho A a}{g_c}\, dV$$

Canceling the area and solving for dV, we have

$$dV = \frac{g_c\,dp}{\rho a} \tag{4.3}$$

Equations (4.2) and (4.3) may now be combined to eliminate dV, with the result

$$a^2 = g_c \frac{dp}{d\rho} \tag{4.4}$$

However, the derivative $dp/d\rho$ is not unique. It depends entirely on the process. Thus it should really be written as a *partial* derivative with the appropriate subscript. But what subscript? What kind of a process are we dealing with?

Remember, we are analyzing an infinitesimal disturbance. For this case we can assume negligible losses and heat transfer as the wave passes through the fluid. Thus the process is both reversible and adiabatic, which means that it is isentropic. (Why?) After we have studied shock waves, we shall prove that very weak shock waves (i.e., small disturbances) approach an isentropic process in the limit. Therefore, equation (4.4) should properly be written as

$$\boxed{a^2 = g_c \left(\frac{\partial p}{\partial \rho}\right)_s} \tag{4.5}$$

This can be expressed in an alternative form by introducing the *bulk* or *volume modulus of elasticity* E_v. This is a relation between volume or density changes that occurs as a result of pressure fluctuations and is defined as

$$E_v \equiv -v\left(\frac{\partial p}{\partial v}\right)_s \equiv \rho\left(\frac{\partial p}{\partial \rho}\right)_s \tag{4.6}$$

Thus

$$\boxed{a^2 = g_c \left(\frac{E_v}{\rho}\right)} \tag{4.7}$$

Equations (4.5) and (4.7) are equivalent general relations for sonic velocity through *any* medium. The bulk modulus is normally used in connection with liquids and solids. Table 4.1 gives some typical values of this modulus, the exact value depending on the temperature and pressure of the medium. For solids it also depends on the type of loading. The reciprocal of the bulk modulus is called the *compressibility*. What is the sonic velocity in a truly incompressible fluid? [*Hint:* What is the value of $(\partial p/\partial \rho)_s$?]

Equation (4.5) is normally used for gases and this can be greatly simplified for the case of a gas that obeys the perfect gas law. For an isentropic process, we know that

Table 4.1 Bulk Modulus Values for Common Media

Medium	Bulk Modulus (psi)
Oil	185,000–270,000
Water	300,000–400,000
Mercury	approx. 4,000,000
Steel	approx. 30,000,000

$$pv^{\gamma} = \text{const} \quad \text{or} \quad p = \rho^{\gamma}\ \text{const} \tag{4.8}$$

Thus

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \rho^{\gamma-1}\ \text{const}$$

But from (4.8), the constant $= p/\rho^{\gamma}$. Therefore,

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \rho^{\gamma-1}\frac{p}{\rho^{\gamma}} = \gamma\frac{p}{\rho} = \gamma RT$$

and from (4.5)

$$a^2 = \gamma g_c RT \tag{4.9}$$

or

$$a = \sqrt{\gamma g_c RT} \tag{4.10}$$

Notice that for perfect gases, sonic velocity is a function of the individual gas and temperature *only*.

Example 4.1 Compute the sonic velocity in air at 70°F.

$$a^2 = \gamma g_c RT = (1.4)(32.2)(53.3)(460 + 70)$$
$$a = 1128 \text{ ft/sec}$$

Example 4.2 Sonic velocity through carbon dioxide is 275 m/s. What is the temperature in Kelvin?

$$a^2 = \gamma g_c RT$$
$$(275)^2 = (1.29)(1)(189)(T)$$
$$T = 310.2 \text{ K}$$

Always keep in mind that in general, sonic velocity is a property of the fluid and varies with the state of the fluid. *Only* for gases that can be treated as perfect is the sonic velocity a function of temperature alone.

Mach Number

We define the *Mach number* as

$$\boxed{M \equiv \frac{V}{a}} \tag{4.11}$$

where

$V \equiv$ the velocity of the medium

$a \equiv$ sonic velocity through the medium

It is important to realize that both V and a are computed *locally* for conditions that actually exist at the same point. If the velocity at one point in a flow system is twice that at another point, we *cannot* say that the Mach number has doubled. We must seek further information on the sonic velocity, which has probably also changed. (What property would we be interested in if the fluid were a perfect gas?)

If the velocity is less than the local speed of sound, M is less than 1 and the flow is called *subsonic*. If the velocity is greater than the local speed of sound, M is greater than 1 and the flow is called *supersonic*. We shall soon see that the Mach number is the most important parameter in the analysis of compressible flows.

4.4 WAVE PROPAGATION

Let us examine a point disturbance that is at rest in a fluid. *Infinitesimal* pressure pulses are continually being emitted and thus they travel through the medium at *sonic* velocity in the form of spherical wave fronts. To simplify matters we shall keep track of only those pulses that are emitted every second. At the end of 3 seconds the picture will appear as shown in Figure 4.3. Note that the wave fronts are concentric.

Now consider a similar problem in which the disturbance is no longer stationary. Assume that it is moving at a speed less than sonic velocity, say $a/2$. Figure 4.4 shows such a situation at the end of 3 seconds. Note that the wave fronts are no longer concentric. Furthermore, the wave that was emitted at $t = 0$ is always in front of the disturbance itself. *Therefore, any person, object, or fluid particle located upstream will feel the wave fronts pass by and know that the disturbance is coming*.

Next, let the disturbance move at exactly sonic velocity. Figure 4.5 shows this case and you will note that all wave fronts coalesce on the left side and move along with the disturbance. After a long period of time this wave front would approximate a plane indicated by the dashed line. In this case, no region upstream is forewarned of the disturbance as the disturbance arrives at the same time as the wave front.

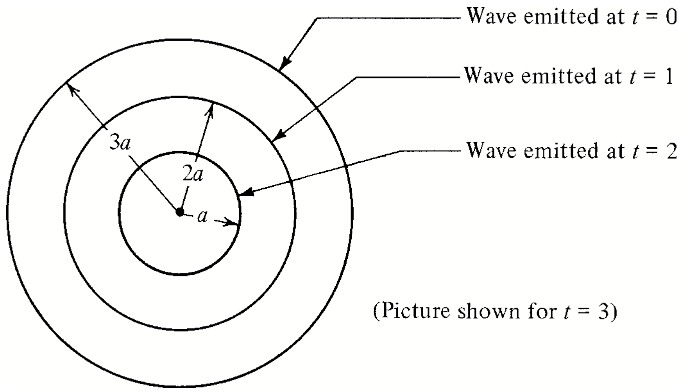

Figure 4.3 Wave fronts from a stationary disturbance.

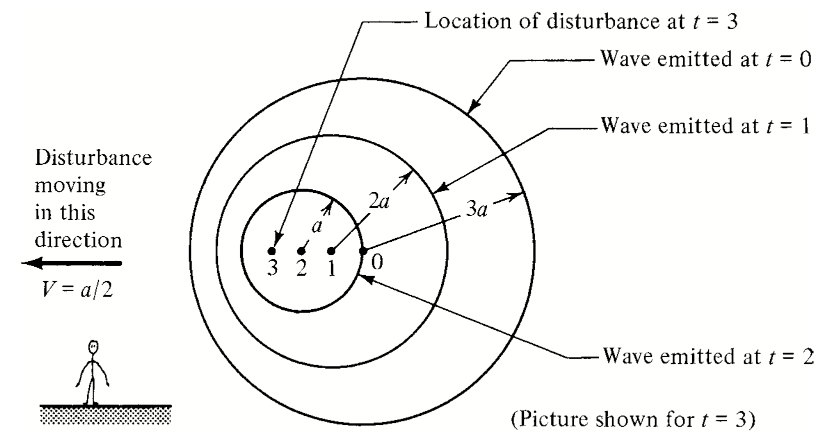

Figure 4.4 Wave fronts from subsonic disturbance.

The only other case to consider is that of a disturbance moving at velocities greater than the speed of sound. Figure 4.6 shows a point disturbance moving at Mach number = 2 (twice sonic velocity). The wave fronts have coalesced to form a cone with the disturbance at the apex. This is called a *Mach cone*. The region inside the cone is called the *zone of action* since it feels the presence of the waves. The outer region is called the *zone of silence*, as *this entire region is unaware of the disturbance*. The surface of the Mach cone is sometimes referred to as a *Mach wave*; the half-angle at the apex is called the *Mach angle* and is given the symbol μ. It should be easy to see that

$$\boxed{\sin \mu = \frac{a}{V} = \frac{1}{M}} \tag{4.12}$$

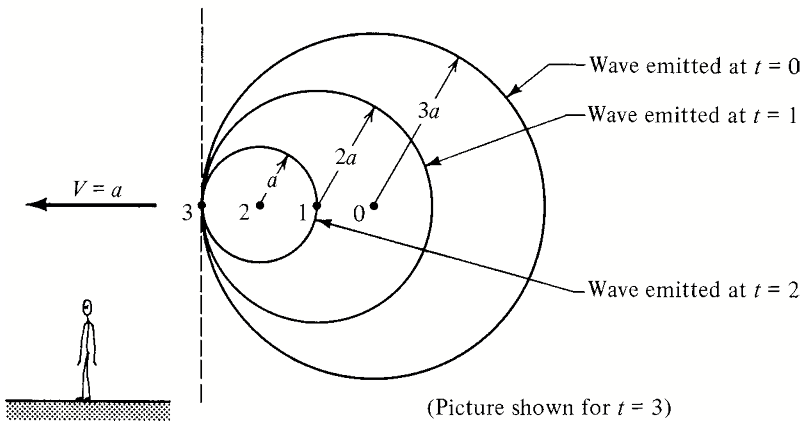

Figure 4.5 Wave fronts from sonic disturbance.

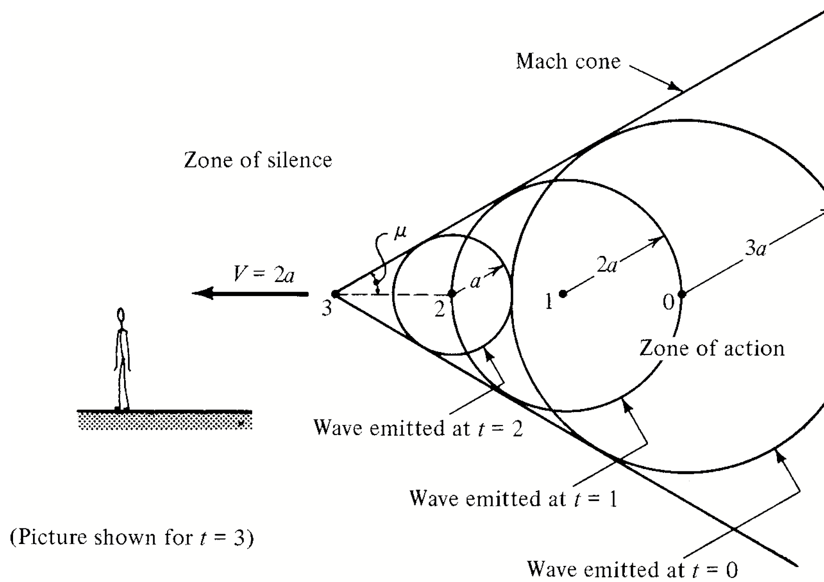

Figure 4.6 Wave fronts from supersonic disturbance.

In this section we have discovered one of the most significant differences between subsonic and supersonic flow fields. In the subsonic case the fluid can "sense" the presence of an object and smoothly adjust its flow around the object. In supersonic flow this is not possible, and thus flow adjustments occur rather abruptly in the form of shock or expansion waves. We study these in great detail in Chapters 6 through 8.

4.5 EQUATIONS FOR PERFECT GASES IN TERMS OF MACH NUMBER

In Section 4.4 we saw that supersonic and subsonic flows have totally different characteristics. This suggests that it would be instructive to use Mach number as a parameter in our basic equations. This can be done very easily for the flow of a perfect gas since in this case we have a simple equation of state *and* an explicit expression for sonic velocity. Development of some of the more important relations follow.

Continuity

For steady one-dimensional flow with a single inlet and a single outlet, we have

$$\dot{m} = \rho A V = \text{ const} \tag{2.30}$$

From the perfect gas equation of state,

$$\rho = \frac{p}{RT} \tag{1.13}$$

and from the definition of Mach number,

$$V = Ma \tag{4.11}$$

Also recall the expression for sonic velocity in a perfect gas:

$$a = \sqrt{\gamma g_c R T} \tag{4.10}$$

Substitution of equations (1.13), (4.11), and (4.10) into (2.30) yields

$$\rho A V = \frac{p}{RT} A M \sqrt{\gamma g_c R T} = pAM\sqrt{\frac{\gamma g_c}{RT}}$$

Thus for steady one-dimensional flow of a perfect gas, the continuity equation becomes

$$\boxed{\dot{m} = pAM\sqrt{\frac{\gamma g_c}{RT}} = \text{ const}} \tag{4.13}$$

Stagnation Relations

For gases we eliminate the potential term and write

$$h_t = h + \frac{V^2}{2g_c} \tag{3.18}$$

Knowing

$$V^2 = M^2 a^2 \qquad \text{[from (4.11)]}$$

and

$$a^2 = \gamma g_c R T \tag{4.9}$$

we have

$$h_t = h + \frac{M^2 \gamma g_c R T}{2 g_c} = h + \frac{M^2 \gamma R T}{2} \tag{4.14}$$

From equations (1.49) and (1.50) we can write the specific heat at constant pressure in terms of γ and R. *Show* that

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{4.15}$$

Combining (4.15) and (4.14), we have

$$h_t = h + M^2 \frac{\gamma - 1}{2} c_p T \tag{4.16}$$

But for a gas we can say that

$$h = c_p T \tag{1.48}$$

Thus

$$h_t = h + M^2 \frac{\gamma - 1}{2} h$$

or

$$\boxed{h_t = h \left(1 + \frac{\gamma - 1}{2} M^2\right)} \tag{4.17}$$

Using $h = c_p T$ and $h_t = c_p T_t$, this can be written as

$$\boxed{T_t = T \left(1 + \frac{\gamma - 1}{2} M^2\right)} \tag{4.18}$$

Equations (4.17) and (4.18) are used frequently. *Memorize them!*

Now, the stagnation process is isentropic. Thus γ can be used as the exponent n in equation (1.57), and between any two points on the same isentropic, we have

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \tag{4.19}$$

Let point 1 refer to the static conditions, and point 2, the stagnation conditions. Then, combining (4.19) and (4.18) produces

$$\frac{p_t}{p} = \left(\frac{T_t}{T}\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} \tag{4.20}$$

or

$$\boxed{p_t = p\left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}} \tag{4.21}$$

This expression for total pressure is important. *Learn it!*

Example 4.3 Air flows with a velocity of 800 ft/sec and has a pressure of 30 psia and temperature of 600°R. Determine the stagnation pressure.

$$a = (\gamma g_c RT)^{1/2} = [(1.4)(32.2)(53.3)(600)]^{1/2} = 1201 \text{ ft/sec}$$

$$M = \frac{V}{a} = \frac{800}{1201} = 0.666$$

$$p_t = p\left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)} = 30\left[1 + \left(\frac{1.4-1}{2}\right)(0.666)^2\right]^{1.4/(1.4-1)}$$

$$p_t = (30)(1 + 0.0887)^{3.5} = (30)(1.346) = 40.4 \text{ psia}$$

Example 4.4 Hydrogen has a static temperature of 25°C and a stagnation temperature of 250°C. What is the Mach number?

$$T_t = T\left(1 + \frac{\gamma-1}{2}M^2\right)$$

$$(250 + 273) = (25 + 273)\left(1 + \frac{1.41-1}{2}M^2\right)$$

$$523 = (298)(1 + 0.205M^2)$$

$$M^2 = 3.683 \quad \text{and} \quad M = 1.92$$

Stagnation Pressure–Energy Equation

For steady one-dimensional flow, we have

$$\frac{dp_t}{\rho_t} + ds_e(T_t - T) + T_t\, ds_i + \delta w_s = 0 \tag{3.25}$$

For a perfect gas,

$$p_t = \rho_t R T_t \tag{4.22}$$

Substitute for the stagnation density and *show* that equation (3.25) can be written as

$$\boxed{\frac{dp_t}{p_t} + \frac{ds_e}{R}\left(1 - \frac{T}{T_t}\right) + \frac{ds_i}{R} + \frac{\delta w_s}{RT_t} = 0} \tag{4.23}$$

A large number of problems are adiabatic and involve no shaft work. In this case, ds_e and δw_s are zero:

$$\frac{dp_t}{p_t} + \frac{ds_i}{R} = 0 \tag{4.24}$$

This can be integrated between two points in the flow system to give

$$\ln \frac{p_{t2}}{p_{t1}} + \frac{s_{i2} - s_{i1}}{R} = 0 \tag{4.25}$$

But since $ds_e = 0$, $ds_i = ds$, and we really do not need to continue writing the subscript i under the entropy. Thus

$$\ln \frac{p_{t2}}{p_{t1}} = -\frac{s_2 - s_1}{R} \tag{4.26}$$

Taking the antilog, this becomes

$$\frac{p_{t2}}{p_{t1}} = e^{-(s_2 - s_1)/R} \tag{4.27}$$

or

$$\boxed{\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R}} \tag{4.28}$$

Watch your units when you use this equation! Total pressures must be absolute, and $\Delta s/R$ must be dimensionless. For this case of adiabatic no-work flow, Δs will always be positive. (Why?) Thus p_{t2} will always be less than p_{t1}. *Only* for the limiting case of no losses will the stagnation pressure remain constant.

This confirms previous knowledge gained from the stagnation pressure–energy equation: that for the case of an adiabatic, no-work system, without flow losses $p_t =$ const for *any* fluid. Thus stagnation pressure is seen to be a very important parameter which in many systems reflects the flow losses. Be careful to note, however, that the specific relation in equation (4.28) is applicable only to perfect gases, and even then only under certain flow conditions. What are these conditions?

Summarizing the above: For steady one-dimensional flow, we have

$$\delta q = \delta w_s + dh_t \tag{3.20}$$

Note that equation (3.20) is valid even if flow losses are present:

$$\text{If } \delta q = \delta w_s = 0, \qquad \text{then} \quad h_t = \text{ constant}$$

If in addition to the above, no losses occur, that is,

$$\text{if } \delta q = \delta w_s = ds_i = 0, \qquad \text{then} \quad p_t = \text{ constant}$$

Example 4.5 Oxygen flows in a constant-area, horizontal, insulated duct. Conditions at section 1 are $p_1 = 50$ psia, $T_1 = 600°$R, and $V_1 = 2860$ ft/sec. At a downstream section the temperature is $T_2 = 1048°$R.

(a) Determine M_1 and T_{t1}.
(b) Find V_2 and p_2.
(c) What is the entropy change between the two sections?

(a) $a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(32.2)(48.3)(600)]^{1/2} = 1143$ ft/sec

$$M_1 = \frac{V_1}{a_1} = \frac{2860}{1143} = 2.50$$

$$T_{t1} = T_1\left(1 + \frac{\gamma - 1}{2} M_1^2\right) = (600)\left[1 + \frac{1.4 - 1}{2}(2.5)^2\right] = 1350°\text{R}$$

(b) *Energy:*

$$h_{t1} + \not{q} = h_{t2} + \not{w}_s$$

$$h_{t1} = h_{t2}$$

and since this is a perfect gas, $T_{t1} = T_{t2}$.

$$T_{t2} = T_2\left(1 + \frac{\gamma - 1}{2} M_2^2\right)$$

$$1350 = (1048)\left(1 + \frac{1.4 - 1}{2} M_2^2\right) \quad \text{and} \quad M_2 = 1.20$$

$$V_2 = M_2 a_2 = (1.20)[(1.4)(32.2)(48.3)(1048)]^{1/2} = 1813 \text{ ft/sec}$$

Continuity:

$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

but

$$A_1 = A_2 \quad \text{and} \quad \rho = p/RT$$

Thus

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

$$p_2 = \frac{V_1}{V_2}\frac{T_2}{T_1} p_1 = \left(\frac{2860}{1813}\right)\left(\frac{1048}{600}\right)(50) = 137.8 \text{ psia}$$

(c) To obtain the entropy change, we need p_{t1} and p_{t2}.

$$p_{t1} = p_1 \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\gamma/(\gamma-1)} = (50)\left[1 + \frac{1.4 - 1}{2}(2.5)^2\right]^{1.4/(1.4-1)} = 854 \text{ psia}$$

Similarly,

$$p_{t2} = 334 \text{ psia}$$

$$e^{-\Delta s/R} = \frac{p_{t2}}{p_{t1}} = \frac{334}{854} = 0.391$$

$$\frac{\Delta s}{R} = \ln \frac{1}{0.391} = 0.939$$

$$\Delta s = \frac{(0.939)(48.3)}{(778)} = 0.0583 \text{ Btu/lbm-°R}$$

4.6 h–s AND T–s DIAGRAMS

Every problem should be approached with a simple sketch of the physical system and also a thermodynamic state diagram. Since the losses affect the entropy changes (through ds_i), one generally uses either an h–s or T–s diagram. In the case of perfect gases, enthalpy is a function of temperature only and therefore the T–s and h–s diagrams are identical except for scale.

Consider a steady one-dimensional flow of a perfect gas. Let us assume no heat transfer and no external work. From the energy equation

$$h_{t1} + \not{q} = h_{t2} + \not{w}_s \tag{3.19}$$

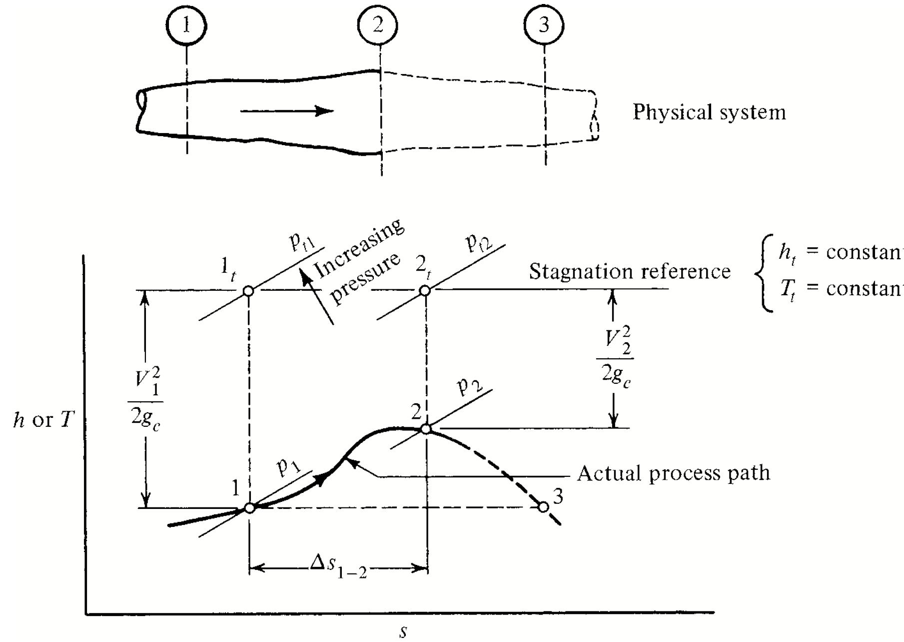

Figure 4.7 Stagnation reference states.

the stagnation enthalpy remains constant, and since it is a perfect gas, the total temperature is also constant. This is represented by the solid horizontal line in Figure 4.7. Two particular sections in the system have been indicated by 1 and 2. The actual process that takes place between these points is indicated on the T–s diagram.

Notice that although the stagnation conditions do not actually exist in the system, they are also shown on the diagram for reference. The distance between the static and stagnation points is indicative of the velocity that exists at that location (since gravity has been neglected). It can also be clearly seen that if there is a Δs_{1-2}, then $p_{t2} < p_{t1}$ and the relationship between stagnation pressure and flow losses is again verified.

It is interesting to hypothesize a third section that just happens to be at the same enthalpy (and temperature) as the first. What else do these points have in common? The same velocity? Obviously! How about sonic velocity? (Recall for gases that this is a function of temperature only.) This means that points 1 and 3 would also have the same Mach number (something that is not immediately obvious). One can now imagine that someplace on this diagram there is a horizontal line that represents the locus of points having a Mach number of unity. Between this line and the stagnation line lie all points in the subsonic regime. Below this line lie all points in the supersonic regime. These conclusions are based on certain assumptions. What are they?

4.7 SUMMARY

In general, waves propagate at a speed that depends on the medium, its thermodynamic state, and the strength of the wave. However, infinitesimal disturbances travel at a speed determined only by the medium and its state. Sound waves fall into this latter category. A discussion of wave propagation and sonic velocity brought out a basic difference between subsonic and supersonic flows. If subsonic, the flow can "sense" objects and flow smoothly around them. This is not possible in supersonic flow, and this topic will be discussed further after the appropriate background has been laid.

As you progress through the remainder of this book and analyze specific flow situations, it will become increasingly evident that fluids behave quite differently in the supersonic regime than they do in the more familiar subsonic flow regime. Thus it will not be surprising to see Mach number become an important parameter. The significance of T–s diagrams as a key to problem visualization should not be overlooked.

Some of the most frequently used equations that were developed in this unit are summarized below. Most are restricted to the steady one-dimensional flow of any fluid, while others apply only to perfect gases. You should determine under what conditions each may be used.

1. *Sonic velocity* (propagation speed of infinitesimal pressure pulses)

$$a^2 = g_c \left(\frac{\partial p}{\partial \rho}\right)_s = g_c \frac{E_v}{\rho} \qquad (4.5), (4.7)$$

$$M = \frac{V}{a} \qquad \text{(all at the same location)} \qquad (4.11)$$

$$\sin \mu = \frac{1}{M} \qquad (4.12)$$

2. *Special relations for perfect gases*

$$a^2 = \gamma g_c R T \qquad (4.9)$$

$$h_t = h\left(1 + \frac{\gamma - 1}{2} M^2\right) \qquad (4.17)$$

$$T_t = T\left(1 + \frac{\gamma - 1}{2} M^2\right) \qquad (4.18)$$

$$p_t = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)} \qquad (4.21)$$

$$\frac{dp_t}{p_t} + \frac{ds_e}{R}\left(1 - \frac{T}{T_t}\right) + \frac{ds_i}{R} + \frac{\delta w_s}{RT_t} = 0 \qquad (4.23)$$

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \qquad \text{for } Q = W = 0 \tag{4.28}$$

PROBLEMS

4.1. Compute and compare sonic velocity in air, hydrogen, water, and mercury. Assume normal room temperature and pressure.

4.2. At what temperature and pressure would carbon monoxide, water vapor, and helium have the same speed of sound as standard air (288 K and 1 atm)?

4.3. Start with the relation for stagnation pressure that is valid for a perfect gas:

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

Expand the right side in a binomial series and evaluate the result for small (but not zero) Mach numbers. Show that your answer can be written as

$$p_t = p + \frac{\rho V^2}{2g_c} + \text{HOT}$$

Remember, the higher-order terms are negligible only for very small Mach numbers. (See Problem 4.4.)

4.4. Measurement of airflow shows the static and stagnation pressures to be 30 and 32 psig, respectively. (Note that these are gage pressures.) Assume that $p_{\text{amb}} = 14.7$ psia and the temperature is 120°F.

(a) Find the flow velocity using equation (4.21).

(b) Now assume that the air is incompressible and calculate the velocity using equation (3.39).

(c) Repeat parts (a) and (b) for static and stagnation pressures of 30 and 80 psig, respectively.

(d) Can you reach any conclusions concerning when a gas may be treated as a constant-density fluid?

4.5. If $\gamma = 1.2$ and the fluid is a perfect gas, what Mach number will give a temperature ratio of $T/T_t = 0.909$? What will the ratio of p/p_t be for this flow?

4.6. Carbon dioxide with a temperature of 335 K and a pressure of 1.4×10^5 N/m^2 is flowing with a velocity of 200 m/s.

(a) Determine the sonic velocity and Mach number.

(b) Determine the stagnation density.

4.7. The temperature of argon is 100°F, the pressure 42 psia, and the velocity 2264 ft/sec. Calculate the Mach number and stagnation pressure.

4.8. Helium flows in a duct with a temperature of 50°C, a pressure of 2.0 bar abs., and a total pressure of 5.3 bar abs. Determine the velocity in the duct.

4.9. An airplane flies 600 mph at an altitude of 16,500 ft, where the temperature is 0°F and the pressure is 1124 psfa. What temperature and pressure might you expect on the nose of the airplane?

4.10. Air flows at $M = 1.35$ and has a stagnation enthalpy of 4.5×10^5 J/kg. The stagnation pressure is 3.8×10^5 N/m^2. Determine the static conditions (pressure, temperature, and velocity).

4.11. A large chamber contains a perfect gas under conditions p_1, T_1, h_1, and so on. The gas is allowed to flow from the chamber (with $q = w_s = 0$). Show that the velocity cannot be greater than

$$V_{\text{max}} = a_1 \left(\frac{2}{\gamma - 1} \right)^{1/2}$$

If the velocity is the maximum, what is the Mach number?

4.12. Air flows steadily in an adiabatic duct where no shaft work is involved. At one section, the total pressure is 50 psia, and at another section, it is 67.3 psia. In which direction is the fluid flowing, and what is the entropy change between these two sections?

4.13. Methane gas flows in an adiabatic, no-work system with negligible change in potential. At one section $p_1 = 14$ bar abs., $T_1 = 500$ K, and $V_1 = 125$ m/s. At a downstream section $M_2 = 0.8$.

(a) Determine T_2 and V_2.

(b) Find p_2 assuming that there are no friction losses.

(c) What is the area ratio A_2/A_1?

4.14. Air flows through a constant-area, insulated passage. Entering conditions are $T_1 = 520$°R, $p_1 = 50$ psia, and $M_1 = 0.45$. At a point downstream, the Mach number is found to be unity.

(a) Solve for T_2 and p_2.

(b) What is the entropy change between these two sections?

(c) Determine the wall frictional force if the duct is 1 ft in diameter.

4.15. Carbon dioxide flows in a horizontal adiabatic, no-work system. Pressure and temperature at section 1 are 7 atm and 600 K. At a downstream section, $p_2 = 4$ atm., $T_2 = 550$ K, and the Mach number is $M_2 = 0.90$.

(a) Compute the velocity at the upstream location.

(b) What is the entropy change?

(c) Determine the area ratio A_2/A_1.

4.16. Oxygen with $T_{t1} = 1000$°R, $p_{t1} = 100$ psia, and $M_1 = 0.2$ enters a device with a cross-sectional area $A_1 = 1$ ft^2 . There is no heat transfer, work transfer, or losses as the gas passes through the device and expands to 14.7 psia.

(a) Compute ρ_1, V_1, and $\dot{m}$.

(b) Compute M_2, T_2, V_2, ρ_2, and A_2.

(c) What force does the fluid exert on the device?

4.17. Consider steady, one-dimensional, constant-area, horizontal, isothermal flow of a perfect gas with no shaft work (Figure P4.17). The duct has a cross-sectional area A and perimeter P. Let τ_w be the shear stress at the wall.

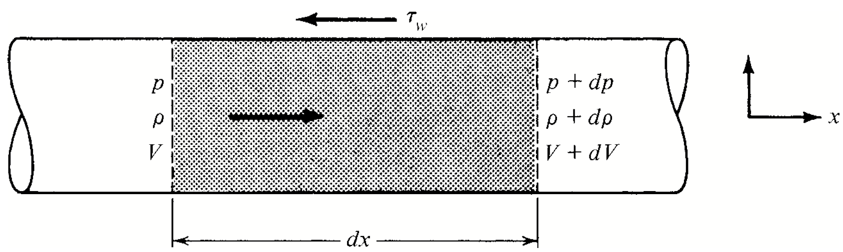

Figure P4.17

(a) Apply momentum concepts [equation (3.45)] and show that

$$-dp - f\frac{dx}{D_e}\frac{\rho V^2}{2g_c} = \frac{\rho V\,dV}{g_c}$$

(b) From the concept of continuity and the equation of state, show that

$$\frac{d\rho}{\rho} = \frac{dp}{p} = -\frac{dV}{V}$$

(c) Combine the results of parts (a) and (b) to show that

$$\frac{d\rho}{\rho} = \left[\frac{\gamma M^2}{2(\gamma M^2 - 1)}\right]\frac{f\,dx}{D_e}$$

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

4.1. **(a)** Define Mach number and Mach angle.

(b) Give an expression that represents sonic velocity in an arbitrary fluid.

(c) Give the relation used to compute sonic velocity in a perfect gas.

4.2. Consider the steady, one-dimensional flow of a perfect gas with heat transfer. The T–s diagram (Figure CT4.2) shows both static and stagnation points at two locations in the system. It is known that $A = B$.

(a) Is heat transferred into or out of the system?

(b) Is $M_2 > M_1$, $M_2 = M_1$, or $M_2 < M_1$?

4.3. State whether each of the following statements is true or false.

(a) Changing the frame of reference (or superposition of a velocity onto an existing flow) does not change the static enthalpy.

(b) Shock waves travel at sonic velocity through a medium.

(c) In general, one can say that flow losses will show up as a decrease in stagnation enthalpy.

(d) The stagnation process is one of constant entropy.

(e) A Mach cone does not exist for subsonic flow.

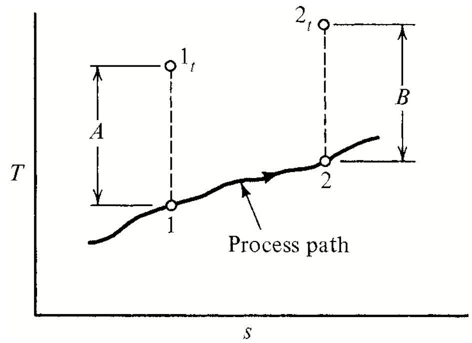

Figure CT4.2

4.4. Cite the conditions that are necessary for the stagnation temperature to remain constant in a flow system.

4.5. For steady flow of a perfect gas, the continuity equation can be written as

$$\dot{m} = f(p, M, T, \gamma, A, R, g_c) = \text{const}$$

Determine the precise function.

4.6. Work Problem 4.14.

Chapter 5

Varying-Area Adiabatic Flow

5.1 INTRODUCTION

Area changes, friction, and heat transfer are the most important factors that affect the properties in a flow system. Although some situations may involve the simultaneous effects of two or more of these factors, the majority of engineering problems are such that *only one of these factors becomes the dominant influence for any particular device*. Thus it is more than academic interest that leads to the separate study of each of the above-mentioned effects. In this manner it is possible to consider only the controlling factor and develop a simple solution that is within the realm of acceptable engineering accuracy.

In this chapter we consider the general problem of varying-area flow under the assumptions of no heat transfer (adiabatic) and no shaft work. We first consider the flow of an arbitrary fluid without losses and determine how its properties are affected by area changes. The case of a perfect gas is then considered and simple working equations developed to aid in the solution of problems with or without flow losses. The latter case (isentropic flow) lends itself to the construction of tables which are used throughout the remainder of the book. The chapter closes with a brief discussion of the various ways in which nozzle and diffuser performance can be represented.

5.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. (*Optional*) Simplify the basic equations for continuity and energy to relate differential changes in density, pressure, and velocity to the Mach number and a differential change in area for steady, one-dimensional flow through a varying-area passage with no losses.

2. Show graphically how pressure, density, velocity, and area vary in steady, one-dimensional, isentropic flow as the Mach number ranges from zero to supersonic values.
3. Compare the function of a nozzle and a diffuser. Sketch physical devices that perform as each for subsonic and supersonic flow.
4. (*Optional*) Derive the working equations for a perfect gas relating property ratios between two points in adiabatic, no-work flow, as a function of the Mach number (M), ratio of specific heats (γ), and change in entropy (Δs).
5. Define the * reference condition and the properties associated with it (i.e., A^*, p^*, T^*, ρ^*, etc.).
6. Express the loss (Δs_i) (between two points in the flow) as a function of stagnation pressures (p_t) *or* reference areas (A^*). Under what conditions are these relations true?
7. State and interpret the relation between stagnation pressure (p_t) and reference area (A^*) for a process between two points in adiabatic no-work flow.
8. Explain how a converging nozzle performs with various receiver pressures. Do the same for the *isentropic* performance of a converging–diverging nozzle.
9. State what is meant by the first and third critical modes of nozzle operation. Given the area ratio of a converging–diverging nozzle, determine the operating pressure ratios that cause operation at the first and third critical points.
10. With the aid of an h–s diagram, give a suitable definition for both nozzle efficiency and diffuser performance.
11. Describe what is meant by a *choked* flow passage.
12. Demonstrate the ability to utilize the adiabatic and isentropic flow relations and the isentropic table to solve typical flow problems.

5.3 GENERAL FLUID-NO LOSSES

We first consider the general behavior of an arbitrary fluid. To isolate the effects of area change, we make the following assumptions:

Steady, one-dimensional flow	
Adiabatic	$\delta q = 0, ds_e = 0$
No shaft work	$\delta w_s = 0$
Neglect potential	$dz = 0$
No losses	$ds_i = 0$

Our objective will be to obtain relations that indicate the variation of fluid properties with area changes *and* Mach number. In this manner we can distinguish the important differences between subsonic and supersonic behavior. We start with the energy equation:

$$\delta q = \delta w_s + dh + \frac{dV^2}{2g_c} + \frac{g}{g_c}\,dz \qquad (2.53)$$

But

$$\delta q = \delta w_s = 0$$

and

$$dz = 0$$

which leaves

$$0 = dh + \frac{dV^2}{2g_c} \qquad (5.1)$$

or

$$dh = -\frac{V\,dV}{g_c} \qquad (5.2)$$

We now introduce the property relation

$$T\,ds = dh - \frac{dp}{\rho} \qquad (1.41)$$

Since our flow situation has been assumed to be adiabatic ($ds_e = 0$) and to contain no losses ($ds_i = 0$), it is also isentropic ($ds = 0$). Thus equation (1.41) becomes

$$dh = \frac{dp}{\rho} \qquad (5.3)$$

We equate equations (5.2) and (5.3) to obtain

$$-\frac{V\,dV}{g_c} = \frac{dp}{\rho}$$

or

$$dV = -\frac{g_c\,dp}{\rho V} \qquad (5.4)$$

We introduce this into equation (2.32) and the differential form of the continuity equation becomes

$$\frac{d\rho}{\rho} + \frac{dA}{A} - \frac{g_c\,dp}{\rho V^2} = 0 \qquad (5.5)$$

Solve this for dp/ρ and *show* that

$$\frac{dp}{\rho} = \frac{V^2}{g_c}\left(\frac{d\rho}{\rho} + \frac{dA}{A}\right) \tag{5.6}$$

Recall the definition of sonic velocity:

$$a^2 = g_c\left(\frac{\partial p}{\partial \rho}\right)_s \tag{4.5}$$

Since our flow *is* isentropic, we may drop the subscript and change the partial derivative to an ordinary derivative:

$$a^2 = g_c\frac{dp}{d\rho} \tag{5.7}$$

This permits equation (5.7) to be rearranged to

$$dp = \frac{a^2}{g_c}\,d\rho \tag{5.8}$$

Substituting this expression for dp into equation (5.6) yields

$$\frac{d\rho}{\rho} = \frac{V^2}{a^2}\left(\frac{d\rho}{\rho} + \frac{dA}{A}\right) \tag{5.9}$$

Introduce the definition of Mach number,

$$M^2 = \frac{V_2^2}{a^2} \tag{4.11}$$

and combine the terms in $d\rho/\rho$ to obtain the following relation between density and area changes:

$$\frac{d\rho}{\rho} = \left(\frac{M^2}{1-M^2}\right)\frac{dA}{A} \tag{5.10}$$

If we now substitute equation (5.10) into the differential form of the continuity equation (2.32), we can obtain a relation between velocity and area changes. *Show* that

$$\frac{dV}{V} = -\left(\frac{1}{1-M^2}\right)\frac{dA}{A} \tag{5.11}$$

Now equation (5.4) can be divided by V to yield

$$\frac{dV}{V} = -\frac{g_c\, dp}{\rho V^2} \tag{5.12}$$

If we equate (5.11) and (5.12), we can obtain a relation between pressure and area changes. *Show* that

$$dp = \frac{\rho V^2}{g_c}\left(\frac{1}{1-M^2}\right)\frac{dA}{A} \tag{5.13}$$

For convenience, we collect the three important relations that will be referred to in the analysis that follows:

$$dp = \frac{\rho V^2}{g_c}\left(\frac{1}{1-M^2}\right)\frac{dA}{A} \tag{5.13}$$

$$\frac{d\rho}{\rho} = \left(\frac{M^2}{1-M^2}\right)\frac{dA}{A} \tag{5.10}$$

$$\frac{dV}{V} = -\left(\frac{1}{1-M^2}\right)\frac{dA}{A} \tag{5.11}$$

Let us consider what is happening as fluid flows through a variable-area duct. For simplicity we shall *assume that the pressure is always decreasing*. Thus dp is negative. From equation (5.13) you see that if $M < 1$, dA must be negative, indicating that the area is decreasing; whereas if $M > 1$, dA must be positive and the area is increasing.

Now continue to assume that the pressure is decreasing. Knowing the area variation you can now consider equation (5.10). Fill in the following blanks with the words *increasing* or *decreasing*: If $M < 1$ (and dA is __________), then $d\rho$ must be __________. If $M > 1$ (and dA is __________), then $d\rho$ must be __________.

Looking at equation (5.11) reveals that if $M < 1$ (and dA is__________) then, dV must be __________ meaning that velocity is __________, whereas if $M > 1$ (and dA is __________), then dV must be __________ and velocity is __________.

We summarize the above by saying that *as the pressure decreases*, the following variations occur:

		Subsonic ($M < 1$)	Supersonic ($M > 1$)
Area	A	Decreases	Increases
Density	ρ	Decreases	Decreases
Velocity	V	Increases	Increases

A similar chart could easily be made for the situation where pressure increases, but it is probably more convenient to express the above in an alternative graphical form, as

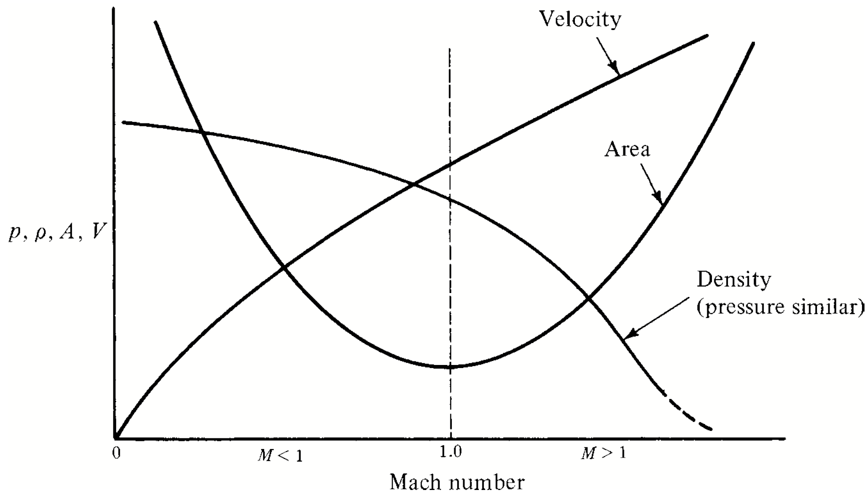

Figure 5.1 Property variation with area change.

shown in Figure 5.1. The appropriate shape of these curves can easily be visualized if one combines equations (5.10) and (5.11) to eliminate the term dA/A with the following result:

$$\frac{d\rho}{\rho} = -M^2 \frac{dV}{V} \tag{5.14}$$

From this equation we see that at low Mach numbers, density variations will be quite small, whereas at high Mach numbers the density changes *very* rapidly. (Eventually, as V becomes very large and ρ becomes very small, small density changes occur once again.) This means that the density is nearly constant in the low subsonic regime ($d\rho \approx 0$) and the velocity changes compensate for area changes. [See the differential form of the continuity equation (2.32).] At a Mach number equal to unity, we reach a situation where density changes and velocity changes compensate for one another and thus no change in area is required ($dA = 0$). As we move on into the supersonic area, the density decreases so rapidly that the accompanying velocity change cannot accommodate the flow and thus the area must increase. We now recognize another aspect of flow behavior which is exactly opposite in subsonic and supersonic flow. Consider the operation of devices such as nozzles and diffusers.

A *nozzle* is a device that converts enthalpy (or pressure energy for the case of an incompressible fluid) into kinetic energy. From Figure 5.1 we see that an increase in velocity is accompanied by either an increase or decrease in area, depending on the Mach number. Figure 5.2 shows what these devices look like in the subsonic and supersonic flow regimes.

A *diffuser* is a device that converts kinetic energy into enthalpy (or pressure energy for the case of incompressible fluids). Figure 5.3 shows what these devices look like

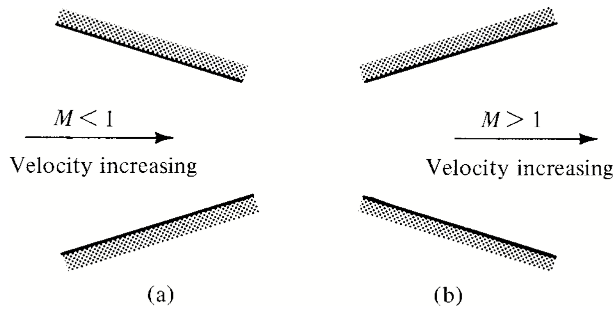

Figure 5.2 Nozzle configurations.

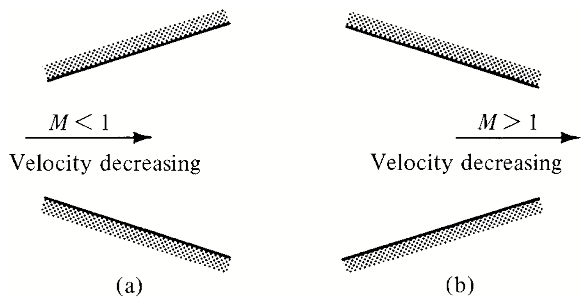

Figure 5.3 Diffuser configurations.

in the subsonic and supersonic regimes. Thus we see that the same piece of equipment can operate as either a nozzle or a diffuser, depending on the flow regime.

Notice that a device is called a nozzle or a diffuser because of *what it does*, not what it looks like. Further consideration of Figures 5.1 and 5.2 leads to some interesting conclusions. If one attached a converging section (see Figure 5.2*a*) to a high-pressure supply, one could never attain a flow greater than Mach 1, regardless of the pressure differential available. On the other hand, if we made a converging–diverging device (combination of Figure 5.2*a* and *b*), we see a means of accelerating the fluid into the supersonic regime, provided that the proper pressure differential exists. Specific examples of these cases are given later in the chapter.

5.4 PERFECT GASES WITH LOSSES

Now that we understand the general effects of area change in a flow system, we will develop some specific working equations for the case of a perfect gas. The term *working equations* will be used throughout this book to indicate relations between properties at arbitrary sections of a flow system written in terms of Mach numbers,

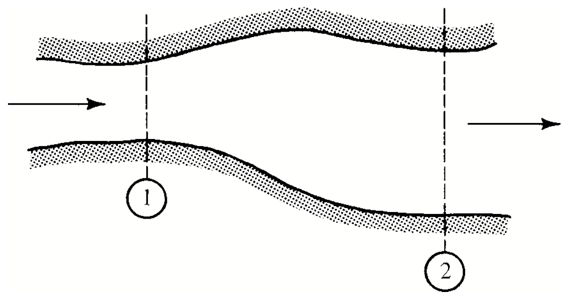

Figure 5.4 Varying-area flow system.

specific heat ratio, and a loss indicator such as Δs_i. An example of this for the system shown in Figure 5.4 is

$$\frac{p_2}{p_1} = f(M_1, M_2, \gamma, \Delta s_i) \tag{5.15}$$

We begin by feeding the following assumptions into our fundamental concepts of state, continuity, and energy:

Steady one-dimensional flow
Adiabatic
No shaft work
Perfect gas
Neglect potential

State

We have the perfect gas equation of state:

$$p = \rho RT \tag{1.13}$$

Continuity

$$\dot{m} = \rho AV = \text{const} \tag{2.30}$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{5.16}$$

We first seek the area ratio

$$\frac{A_2}{A_1} = \frac{\rho_1 V_1}{\rho_2 V_2} \tag{5.17}$$

We substitute for the densities using the equation of state (1.13) and for velocities from the definition of Mach number (4.11):

$$\frac{A_2}{A_1} = \left(\frac{p_1}{RT_1}\right)\left(\frac{RT_2}{p_2}\right)\frac{M_1 a_1}{M_2 a_2} = \frac{p_1 T_2 M_1 a_1}{p_2 T_1 M_2 a_2} \tag{5.18}$$

Introduce the expression for the sonic velocity of a perfect gas:

$$a = \sqrt{\gamma g_c RT} \tag{4.10}$$

and *show* that equation (5.18) becomes

$$\frac{A_2}{A_1} = \frac{p_1 M_1}{p_2 M_2}\left(\frac{T_2}{T_1}\right)^{1/2} \tag{5.19}$$

We must now find a means to express the pressure and temperature ratios in terms of M_1, M_2, γ, and Δs.

Energy

We start with

$$h_{t1} + q = h_{t2} + w_s \tag{3.19}$$

For an adiabatic, no-work process, this shows that

$$h_{t1} = h_{t2} \tag{5.20}$$

However, we can go further than this since we know that for a perfect gas, enthalpy is a function of temperature *only*. Thus

$$T_{t1} = T_{t2} \tag{5.21}$$

Recall from Chapter 4 that we developed a general relationship between static and stagnation temperatures for a perfect gas as

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.18}$$

Hence equation (5.21) can be written as

$$T_1\left(1 + \frac{\gamma - 1}{2}M_1^2\right) = T_2\left(1 + \frac{\gamma - 1}{2}M_2^2\right) \tag{5.22}$$

or

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2} \tag{5.23}$$

which is the ratio desired for equation (5.19). Note that no subscripts have been put on the specific heat ratio γ, which means we are assuming that $\gamma_1 = \gamma_2$. This might be questioned since the specific heats c_p and c_v are known to vary somewhat with temperature. In Chapter 11 we explore real gas behavior and learn why these specific heats vary and discover that their *ratio* (γ) does not exhibit much change except over large temperature ranges. Thus the assumption of constant γ generally leads to acceptable engineering accuracy.

Recall from Chapter 4 that we also developed a general relationship between static and stagnation pressures for a perfect gas:

$$p_t = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)} \tag{4.21}$$

Furthermore, the stagnation pressure–energy equation was easily integrated for the case of a perfect gas in adiabatic, no-work flow to yield

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

If we introduce equation (4.21) into (4.28), we have

$$\frac{p_{t2}}{p_{t1}} = \frac{p_2}{p_1}\left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{\gamma/(\gamma-1)} = e^{-\Delta s/R} \tag{5.24}$$

Rearrange this to obtain the ratio

$$\frac{p_1}{p_2} = \left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{\gamma/(\gamma-1)} e^{+\Delta s/R} \tag{5.25}$$

We now have the desired information to accomplish the original objective. Direct substitution of equations (5.23) and (5.25) into (5.19) yields

$$\frac{A_2}{A_1} = \left[\left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{\gamma/(\gamma-1)} e^{\Delta s/R}\right] \times \frac{M_1}{M_2}\left(\frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2}\right)^{1/2} \tag{5.26}$$

Show that this can be simplified to

$$\boxed{\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{(\gamma+1)/2(\gamma-1)} e^{\Delta s/R}} \tag{5.27}$$

Note that to obtain this equation, we automatically discovered a number of other working equations, which for convenience we summarize below.

$$T_{t1} = T_{t2} \tag{5.21}$$

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^{\,2}}{1 + [(\gamma - 1)/2]M_2^{\,2}} \tag{5.23}$$

$$\frac{p_2}{p_1} = \left(\frac{1 + [(\gamma - 1)/2]M_1^{\,2}}{1 + [(\gamma - 1)/2]M_2^{\,2}}\right)^{\gamma/(\gamma-1)} e^{-\Delta s/R} \qquad \text{from} \qquad (5.25)$$

From equations (1.13), (5.23), and (5.25) you should also be able to show that

$$\frac{\rho_2}{\rho_1} = \left(\frac{1 + [(\gamma - 1)/2]M_1^{\,2}}{1 + [(\gamma - 1)/2]M_2^{\,2}}\right)^{1/(\gamma-1)} e^{-\Delta s/R} \tag{5.28}$$

Example 5.1 Air flows in an adiabatic duct without friction. At one section the Mach number is 1.5, and farther downstream it has increased to 2.8. Find the area ratio.

For a frictionless, adiabatic system, $\Delta s = 0$. We substitute directly into equation (5.27):

$$\frac{A_2}{A_1} = \frac{1.5}{2.8}\left[\frac{1 + [(1.4 - 1)/2](2.8)^2}{1 + [(1.4 - 1)/2](1.5)^2}\right]^{(1.4+1)/2(1.4-1)} (1) = 2.98$$

This problem is very simple since both Mach numbers are known. The inverse problem (given A_1, A_2, and M_1, find M_2) is not so straightforward. We shall come back to this in Section 5.6 after we develop a new concept.

5.5 THE * REFERENCE CONCEPT

In Section 3.5 the concept of a stagnation reference state was introduced, which by the nature of its definition turned out to involve an isentropic process. Before going any further with the working equations developed in Section 5.4, it will be convenient to introduce another reference condition because, among other things, the stagnation state is not a feasible reference when dealing with area changes. (Why?) We denote this new reference state with a superscript * and define it as "that thermodynamic state which would exist if the fluid reached a Mach number of unity *by some particular process*". The italicized phrase is significant, for there are many processes by which we could reach Mach 1.0 from any given starting point, and they would each lead to a different thermodynamic state. Every time we analyze a different flow phenomenon we will be considering different types of processes, and thus we will be dealing with a different * reference state.

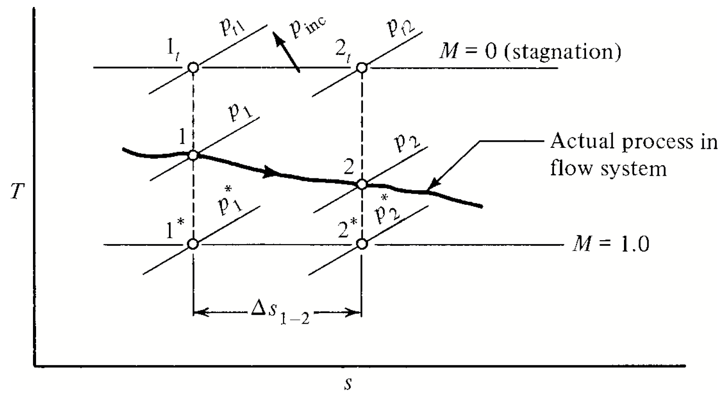

Figure 5.5 Isentropic * reference states.

We first consider a * reference state reached under reversible-adiabatic conditions (i.e., by an isentropic process). Every point in the flow system has its own * reference state, just as it has its own stagnation reference state. As an illustration, consider a system that involves the flow of a perfect gas with no heat or work transfer. Figure 5.5 shows a *T–s* diagram indicating two points in such a flow system. Above each point is shown its stagnation reference state, and we now add the isentropic * reference state that is associated with each point. Not only is the stagnation line for the entire system a horizontal line, but in this system all * reference points will lie on a horizontal line (see the discussion in Section 4.6). Is the flow subsonic or supersonic in the system depicted in Figure 5.5?

We now proceed to develop an extremely important relation. Keep in mind that * reference states probably don't exist in the system, but with appropriate area changes *they could exist*, and as such they represent legitimate section locations to be used with any of the equations that we developed earlier [such as equations (5.23), (5.25), (5.27), etc.]. Specifically, let us consider

$$\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_2^{\,2}}{1+[(\gamma-1)/2]M_1^{\,2}}\right)^{(\gamma+1)/2(\gamma-1)} e^{\Delta s/R} \tag{5.27}$$

In this equation, points 1 and 2 represent *any* two points that could exist in a system (subject to the same assumptions that led to the development of the equation). We now apply equation (5.27) between points 1^* and 2^*. Thus

$$A_1 \Rightarrow A_1^* \qquad M_1 \Rightarrow M_1^* \equiv 1$$
$$A_2 \Rightarrow A_2^* \qquad M_2 \Rightarrow M_2^* \equiv 1$$

and we have:

$$\frac{A_2^*}{A_1^*} = \frac{1}{1}\left(\frac{1 + [(\gamma - 1)/2]1^2}{1 + [(\gamma - 1)/2]1^2}\right)^{(\gamma+1)/2(\gamma-1)} e^{\Delta s/R}$$

or

$$\boxed{\frac{A_2^*}{A_1^*} = e^{\Delta s/R}} \tag{5.29}$$

Before going further, it might be instructive to check this relation to see if it appears reasonable. First, take the case of no losses where $\Delta s = 0$. Then equation (5.29) says that $A_1^* = A_2^*$. Check Figure 5.5 for the case of $\Delta s_{1-2} = 0$. Under these conditions the diagram collapses into a single isentropic line on which 1_t is identical with 2_t and 1^* is the same point as 2^*. Under this condition, it should be obvious that A_1^* is the same as A_2^*.

Next, take the more general case where Δs_{1-2} is nonzero. Assuming that these points exist in a flow system, they must pass the same amount of fluid, or

$$\dot{m} = \rho_1^* A_1^* V_1^* = \rho_2^* A_2^* V_2^* \tag{5.30}$$

Recall from Section 4.6 that since these state points are on the same horizontal line,

$$V_1^* = V_2^* \tag{5.31}$$

Similarly, we know that $T_1^* = T_2^*$, and from Figure 5.5 it is clear that $p_1^* > p_2^*$. Thus from the equation of state, we can easily determine that

$$\rho_2^* < \rho_1^* \tag{5.32}$$

Introduce equations (5.31) and (5.32) into (5.30) and *show* that for the case of $\Delta s_{1-2} > 0$,

$$A_2^* > A_1^* \tag{5.33}$$

which agrees with equation (5.29).

We have previously developed a relation between the stagnation pressures (which involves the same assumptions as equation (5.29):

$$\boxed{\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R}} \tag{4.28}$$

Check Figure 5.5 to convince yourself that this equation also appears to give reasonable answers for the special case of $\Delta s = 0$ and for the general case of $\Delta s > 0$.

We now multiply equation (5.29) by equation (4.28):

$$\frac{A_2^*}{A_1^*}\frac{p_{t2}}{p_{t1}} = \left(e^{\Delta s/R}\right)\left(e^{-\Delta s/R}\right) = 1 \tag{5.34}$$

or

$$\boxed{p_{t1}A_1^* = p_{t2}A_2^*} \tag{5.35}$$

This is a most important relation that is frequently the key to problem solutions in adiabatic flow. Learn equation (5.35) and the conditions under which it applies.

5.6 ISENTROPIC TABLE

In Section 5.4 we considered the steady, one-dimensional flow of a perfect gas under the conditions of no heat and work transfer and negligible potential changes. Looking back over the working equations that were developed reveals that many of them do not include the loss term (Δs_i). In those where the loss term does appear, it takes the form of a simple multiplicative factor such as $e^{\Delta s/R}$. This leads to the natural use of the isentropic process as a standard for ideal performance with appropriate corrections made to account for losses when necessary. In a number of cases, we find that some actual processes are so efficient that they are very nearly isentropic and thus need no corrections.

If we simplify equation (5.27) for an isentropic process, it becomes

$$\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{1+[(\gamma-1)/2]M_2^{\,2}}{1+[(\gamma-1)/2]M_1^{\,2}}\right)^{(\gamma+1)/2(\gamma-1)} \tag{5.36}$$

This is easy to solve for the area ratio if both Mach numbers are known (see Example 5.1), but let's consider a more typical problem. The physical situation is fixed (i.e., A_1 and A_2 are known). The fluid (and thus γ) is known, and the Mach number at one location (say, M_1) is known. Our problem is to solve for the Mach number (M_2) at the other location. Although this is not impossible, it is messy and a lot of work.

We can simplify the solution by the introduction of the * reference state. Let point 2 be an arbitrary point in the flow system, and let its isentropic * point be point 1. Then

$$A_2 \Rightarrow A \qquad M_2 \Rightarrow M \quad \text{(any value)}$$
$$A_1 \Rightarrow A^* \qquad M_1 \Rightarrow 1$$

and equation (5.36) becomes

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M,\gamma) \tag{5.37}$$

We see that $A/A^* = f(M,\gamma)$, and we can easily construct a table giving values of A/A^* versus M for a particular γ. The problem posed earlier could then be solved as follows:

Given: γ, A_1, A_2, M_1, and isentropic flow.
Find: M_2.

We approach the solution by formulating the ratio A_2/A_2^* in terms of known quantities.

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} \tag{5.38}$$

- $\frac{A_2}{A_1}$: Given
- $\frac{A_1}{A_1^*}$: A function of M_1; look up in isentropic table
- $\frac{A_1^*}{A_2^*}$: Evaluated by equation (5.29) and equals 1.0 if flow is isentropic

Thus A_2/A_2^* can be calculated, and by entering the isentropic table with this value, M_2 can be determined. *A word of caution here!* The value of A_2/A_2^* will be found in *two* places in the table, as we are really solving equation (5.36), or the more general case equation (5.27), which is a quadratic for M_2. One value will be in the subsonic region and the other in the supersonic regime. You should have no difficulty determining which answer is correct when you consider the physical appearance of the system together with the concepts developed in Section 5.3.

Note that the general problem *with losses* can also be solved by the same technique as long as information is available concerning the loss. This could be given to us in the form of A_1^*/A_2^*, p_{t2}/p_{t1}, or possibly as Δs_{1-2}. All three of these represent equivalent ways of expressing the loss [through equations (4.28) and (5.29)].

We now realize that the key to simplified problem solution is to have available a table of property ratios as a function of γ and *one* Mach number only. These are obtained by taking the equations developed in Section 5.4 and introducing a reference state, either the * reference condition (reached by an isentropic process) or the stagnation reference condition (reached by an isentropic process). We proceed with equation (5.23):

$$\frac{T_2}{T_1} = \frac{1+[(\gamma-1)/2]M_1^2}{1+[(\gamma-1)/2]M_2^2} \tag{5.23}$$

Let point 2 be any arbitrary point in the system and let its stagnation point be point 1. Then

$$T_2 \Rightarrow T \qquad M_2 \Rightarrow M \quad \text{(any value)}$$

$$T_1 \Rightarrow T_t \qquad M_1 \Rightarrow 0$$

and equation (5.23) becomes

$$\frac{T}{T_t} = \frac{1}{1 + [(\gamma - 1)/2]M^2} = f(M, \gamma) \tag{5.39}$$

Equation (5.25) can be treated in a similar fashion. In this case we let 1 be the arbitrary point and its stagnation point is taken as 2. Then

$$p_1 \Rightarrow p \qquad M_1 \Rightarrow M \quad \text{(any value)}$$

$$p_2 \Rightarrow p_t \qquad M_2 \Rightarrow 0$$

and when we remember that the stagnation process is isentropic, equation (5.25) becomes

$$\frac{p}{p_t} = \left(\frac{1}{1 + [(\gamma - 1)/2]M^2}\right)^{\gamma/(\gamma-1)} = f(M, \gamma) \tag{5.40}$$

Equations (5.39) and (5.40) are not surprising, as we have developed these previously by other methods [see equations (4.18) and (4.21)]. The tabulation of equation (5.40) may be used to solve problems in the same manner as the area ratio. For example, assume that we are

Given: γ, p_1, p_2, M_2, and Δs_{1-2} and asked to
Find: M_1.

To solve this problem, we seek the ratio p_1/p_{t1} in terms of known ratios:

$$\frac{p_1}{p_{t1}} = \frac{p_1}{p_2}\frac{p_2}{p_{t2}}\frac{p_{t2}}{p_{t1}} \tag{5.41}$$

- p_1/p_2: Given
- p_2/p_{t2}: A function of M_2; look up in isentropic table
- p_{t2}/p_{t1}: Evaluated by equation (4.28) as a function of Δs_{1-2}

After calculating the value of p_1/p_{t1}, we enter the isentropic table and find M_1. Note that even though the flow from station 1 to 2 is *not* isentropic, the functions for p_1/p_{t1} and p_2/p_{t2} *are isentropic by definition*; thus the isentropic table can be used to solve this problem. The connection *between* the two points is made through p_{t2}/p_{t1}, which involves the entropy change.

We could continue to develop other isentropic relations as functions of the Mach number and γ. *Apply* the previous techniques to equation (5.28) and show that

$$\frac{\rho}{\rho_t} = \left(\frac{1}{1 + [(\gamma - 1)/2]M^2}\right)^{1/(\gamma-1)} \tag{5.42}$$

Another interesting relationship is the product of equations (5.37) and (5.40):

$$\frac{A}{A^*}\frac{p}{p_t} = f(M, \gamma) \tag{5.43}$$

Determine what unique function of M and γ is represented in equation (5.43). Since A/A^* and p/p_t are isentropic by definition, we should not be surprised that their product is listed in the isentropic table. But can these functions provide the connection *between* two locations in a flow system *with known losses*?

Recall that

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

and

$$\frac{A_2^*}{A_1^*} = e^{\Delta s/R} \tag{4.29}$$

Thus, for cases involving losses (Δs), changes in A^* are exactly compensated for by changes in p_t. This is true for all steady, one-dimensional flows of a perfect gas in an adiabatic no-work system. We shall see later that equation (5.43) provides the only direct means of solving certain types of problems.

Values of these isentropic flow parameters have been calculated from equations (5.37), (5.39), (5.40), and so on, and tabulated in Appendix G. To convince yourself that there is nothing magical about this table, you might want to check some of the numbers found in them opposite a particular Mach number. In fact, as an exercise in programming a digital computer, you could work up your own set of tables for values of γ other than 1.4, which is the only one included in Appendix G (see Problem 5.24). In Section 5.10 we suggest alternatives to the use of the table. As you read the following examples, look up the numbers in the isentropic table to convince yourself that you know how to find them.

Example 5.2 You are now in a position to rework Example 5.1 with a minimum of calculation. Recall that $M_1 = 1.5$ and $M_2 = 2.8$.

$$\frac{A_2}{A_1} = \frac{A_2}{A_2^*}\frac{A_2^*}{A_1^*}\frac{A_1^*}{A_1} = (3.5001)(1)\left(\frac{1}{1.1762}\right) = 2.98$$

The following information (and Figure E5.3) are common to Examples 5.3 through 5.5. We are given the steady, one-dimensional flow of air ($\gamma = 1.4$), which can be treated as a perfect gas. Assume that $Q = W_s = 0$ and negligible potential changes. $A_1 = 2.0\ \text{ft}^2$ and $A_2 = 5.0\ \text{ft}^2$.

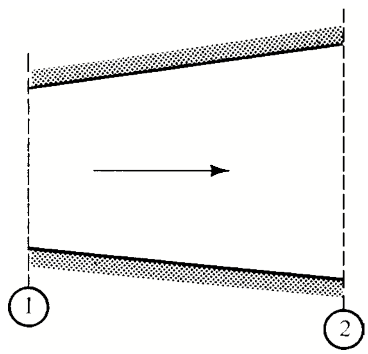

Figure E5.3

Example 5.3 Given that $M_1 = 1.0$ and $\Delta s_{1-2} = 0$. Find the possible values of M_2.

To determine conditions at section 2 in Figure E5.3, we establish the ratio

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{5}{2}\right)(1.000)(1) = 2.5$$

- A_1^*/A_2^*: Equals 1.0 since isentropic
- A_1/A_1^*: From isentropic table at $M = 1.0$
- A_2/A_1: From given physical configuration

Look up $A/A^* = 2.5$ in the isentropic table and determine that $M_2 = 0.24$ or 2.44. We can't tell which Mach number exists without additional information.

Example 5.4 Given that $M_1 = 0.5$, $p_1 = 4$ bar, and $\Delta s_{1-2} = 0$, find M_2 and p_2.

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{5}{2}\right)(1.3398)(1) = 3.35$$

Thus $M_2 \approx 0.175$. (Why isn't it 2.75?)

$$p_2 = \frac{p_2}{p_{t2}}\frac{p_{t2}}{p_{t1}}\frac{p_{t1}}{p_1}p_1 = (0.9788)(1)\left(\frac{1}{0.8430}\right)(4) = 4.64 \text{ bar}$$

Example 5.5 Given: $M_1 = 1.5$, $T_1 = 70°\text{F}$, and $\Delta s_{1-2} = 0$,
Find: M_2 and T_2.

Find $A_2/A_2^* = ?$ (Thus $M_2 \approx 2.62$.)

Once M_2 is known, we can find T_2.

$$T_2 = \frac{T_2}{T_{t2}}\frac{T_{t2}}{T_{t1}}\frac{T_{t1}}{T_1}T_1 = (0.4214)(1)\left(\frac{1}{0.6897}\right)(530) = 324°\text{R}$$

Why is $T_{t1} = T_{t2}$? (Write an energy equation between 1 and 2.)

Example 5.6 Oxygen flows into an insulated device with the following initial conditions: $p_1 = 20$ psia, $T_1 = 600°\text{R}$, and $V_1 = 2960$ ft/sec. After a short distance the area has converged

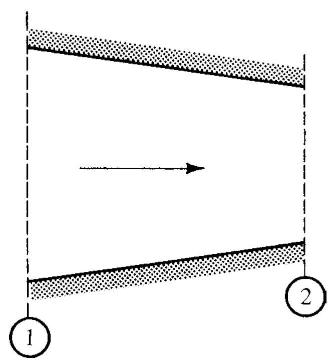

Figure E5.6

from 6 ft^2 to 2.5 ft^2 (Figure E5.6). You may assume steady, one-dimensional flow and a perfect gas. (See the table in Appendix A for gas properties.)

(a) Find M_1, p_{t1}, T_{t1}, and h_{t1}.
(b) If there are losses such that $\Delta s_{1-2} = 0.005$ Btu/1bm-°R, find M_2, p_2, and T_2.

(a) First, we determine conditions at station 1.

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(32.2)(48.3)(600)]^{1/2} = 1143 \text{ ft/sec}$$

$$M_1 = \frac{V_1}{a_1} = \frac{2960}{1143} = 2.59$$

$$p_{t1} = \frac{p_{t1}}{p_1} p_1 = \left(\frac{1}{0.0509}\right)(20) = 393 \text{ psia}$$

$$T_{t1} = \frac{T_{t1}}{T_1} T_1 = \left(\frac{1}{0.4271}\right)(600) = 1405°\text{R}$$

$$h_{t1} = c_p T_{t1} = (0.218)(1405) = 306 \text{ Btu/lbm}$$

(b) For a perfect gas with $q = w_s = 0$, $T_{t1} = T_{t2}$ (from an energy equation), and also from equation (5.29):

$$\frac{A_1^*}{A_2^*} = e^{-\Delta s/R} = e^{-(0.005)(778)/48.3} = 0.9226$$

Thus

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{2.5}{6}\right)(2.8688)(0.9226) = 1.1028$$

From the isentropic table we find that $M_2 \approx$ ______. Why is the use of the isentropic table legitimate here when there are losses in the flow? Continue and compute p_2 and T_2.

$$p_2 = \qquad\qquad (P_2 \approx 117 \text{ psia})$$

$$T_2 = \qquad\qquad (T_2 \approx 1017°\text{R})$$

Could you find the velocity at section 2?

5.7 NOZZLE OPERATION

We will now start a discussion of nozzle operation and at the same time gain more experience in use of the isentropic table. Two types of nozzles are considered: a converging-only nozzle and a converging–diverging nozzle. We start by examining the physical situation shown in Figure 5.6. A source of air at 100 psia and 600°R is contained in a large tank where stagnation conditions prevail. Connected to the tank is a converging-only nozzle and it exhausts into an extremely large receiver where the pressure can be regulated. We can neglect frictional effects, as they are very small in a converging section.

If the receiver pressure is set at 100 psia, no flow results. Once the receiver pressure is lowered below 100 psia, air will flow from the supply tank. Since the supply tank has a large cross section relative to the nozzle outlet area, the velocities in the tank may be neglected. Thus $T_1 \approx T_{t1}$ and $p_1 \approx p_{t1}$. There is no shaft work and we assume no heat transfer. We identify section 2 as the nozzle outlet.

Energy

$$h_{t1} + \not{q} = h_{t2} + \not{w}_s \tag{3.19}$$

$$h_{t1} = h_{t2}$$

and since we can treat this as a perfect gas,

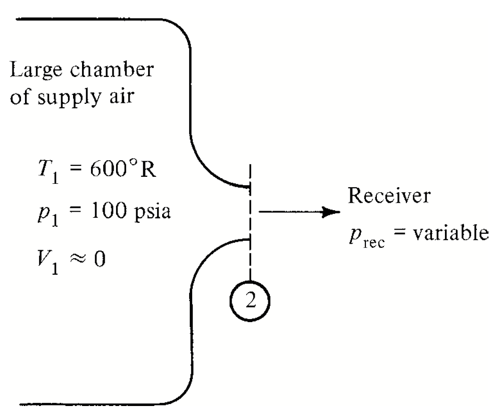

Figure 5.6 Converging-only nozzle.

$$T_{t1} = T_{t2}$$

It is important to recognize that the receiver pressure is controlling the flow. The velocity will increase and the pressure will decrease as we progress through the nozzle until the pressure at the nozzle outlet equals that of the receiver. This will always be true *as long as* the nozzle outlet can "sense" the receiver pressure. Can you think of a situation where pressure pulses from the receiver could not be "felt" inside the nozzle? (Recall Section 4.4.)

Let us assume that

$$p_{\text{rec}} = 80.2 \text{ psia}$$

Then

$$p_2 = p_{\text{rec}} = 80.2 \text{ psia}$$

and

$$\frac{p_2}{p_{t2}} = \frac{p_2}{p_{t1}} \frac{p_{t1}}{p_{t2}} = \left(\frac{80.2}{100}\right)(1) = 0.802$$

Note that $p_{t1} = p_{t2}$ by equation (4.28) since we are neglecting friction.

From the isentropic table corresponding to $p/p_t = 0.802$, we see that

$$M_2 = 0.57 \quad \text{and} \quad \frac{T_2}{T_{t2}} = 0.939$$

Thus

$$T_2 = \left(\frac{T_2}{T_{t2}}\right) T_{t2} = (0.939)(600) = 563°\text{R}$$

$$a_2^2 = (1.4)(32.2)(53.3)(563)$$

$$a_2 = 1163 \text{ ft/sec}$$

and

$$V_2 = M_2 a_2 = (0.57)(1163) = 663 \text{ ft/sec}$$

Figure 5.7 shows this process on a T–s diagram as an isentropic expansion. If the pressure in the receiver were lowered further, the air would expand to this lower pressure and the Mach number and velocity would increase. Assume that the receiver pressure is lowered to 52.83 psia. *Show* that

$$\frac{p_2}{p_{t2}} = 0.5283$$

and thus

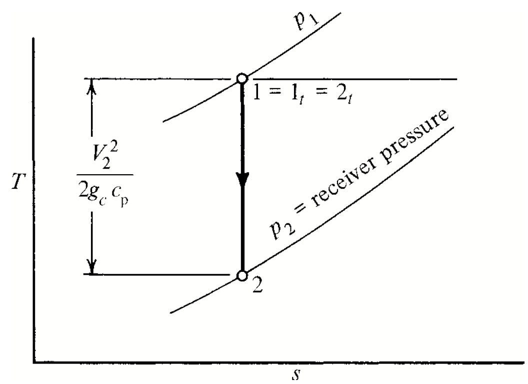

Figure 5.7 T–s diagram for converging-only nozzle.

$$M_2 = 1.00 \qquad \text{with} \quad V_2 = 1096 \text{ ft/sec}$$

Notice that the air velocity coming out of the nozzle is exactly sonic. If we now drop the receiver pressure below this *critical pressure* (52.83 psia), the nozzle has no way of adjusting to these conditions. Why not? Assume that the nozzle outlet pressure could continue to drop along with the receiver. This would mean that $p_2/p_{t2} < 0.5283$, which corresponds to a supersonic velocity. We know that if the flow is to go supersonic, the area must reach a minimum and then increase (see Section 5.3). Thus for a converging-*only* nozzle, the flow is governed by the receiver pressure until sonic velocity is reached at the nozzle outlet and *further reduction of the receiver pressure will have no effect on the flow conditions inside the nozzle*. Under these conditions, the nozzle is said to be *choked* and the nozzle outlet pressure remains at the *critical pressure*. Expansion to the receiver pressure takes place *outside* the nozzle.

In reviewing this example you should realize that there is nothing magical about a receiver pressure of 52.83 psia. The significant item is the *ratio* of the static to total pressure at the exit plane, which for the case of no losses is the *ratio* of the receiver pressure to the inlet pressure. With sonic velocity at the exit, this *ratio* is 0.5283.

The analysis above assumes that conditions within the supply tank remain constant. One should realize that the choked flow rate can change if, for example, the supply pressure or temperature is changed or the size of the throat (exit hole) is changed. It is instructive to take an alternative view of this situation. You are asked in Problem 5.9 to develop the following equation for isentropic flow:

$$\frac{\dot{m}}{A} = M\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-(\gamma+1)/2(\gamma-1)} \left(\frac{\gamma g_c}{R}\right)^{1/2} \frac{p_t}{\sqrt{T_t}} \tag{5.44a}$$

Applying this equation to the outlet and considering choked flow, $M = 1$ and $A = A^*$. Then

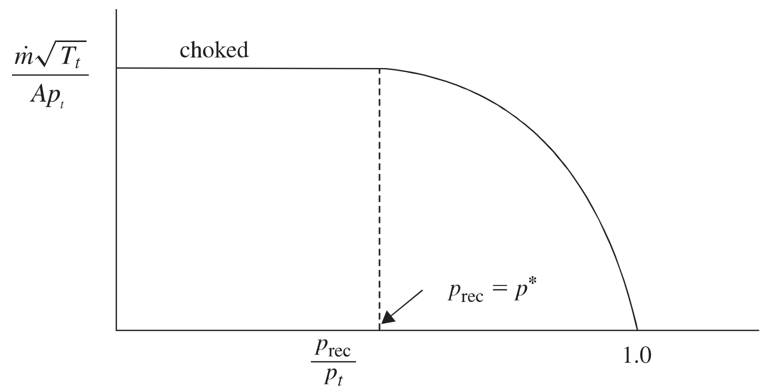

Figure 5.8 Operation of a converging-only nozzle at various back pressures.

$$\left(\frac{\dot{m}}{A}\right)_{max} = \frac{\dot{m}}{A^*} = \left[\frac{\gamma g_c}{R}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\right]^{1/2} \frac{p_t}{\sqrt{T_t}} \qquad (5.44b)$$

For a given gas,

$$\frac{\dot{m}}{A^*} = \text{constant}\ \frac{p_t}{\sqrt{T_t}} \qquad (5.44c)$$

We now look at four distinct possibilities:

1. For a fixed T_t, p_t, and A^* $\Rightarrow$ $\dot{m}_{max}$ constant.
2. For only p_t increasing $\Rightarrow$ $\dot{m}_{max}$ increases.
3. For only T_t increasing $\Rightarrow$ $\dot{m}_{max}$ decreases.
4. For only A^* increasing $\Rightarrow$ $\dot{m}_{max}$ increases.

Figure 5.8 shows this in yet another way.

Converging–Diverging Nozzle

Now let us examine a similar situation but with a converging–diverging nozzle (sometimes called a *DeLaval nozzle*), shown in Figures 5.9 and 5.10. We identify the *throat* (or section of minimum area) as 2 and the exit section as 3. The distinguishing physical characteristic of this type of nozzle is the *area ratio*, meaning the ratio of the exit area to the throat area. Assume this to be $A_3/A_2 = 2.494$. Keep in mind that the objective of making a converging–diverging nozzle is to obtain supersonic flow. Let us first examine the *design operating condition* for this nozzle. If the nozzle is to operate as desired, we know (see Section 5.3) that the flow will be subsonic from 1 to 2, sonic at 2, and supersonic from 2 to 3.

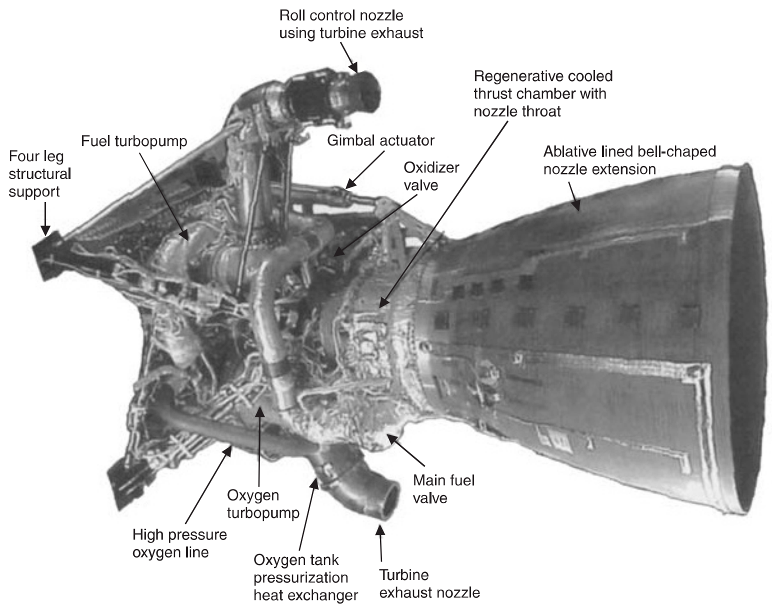

Figure 5.9 Typical converging–diverging nozzle. (Courtesy of the Boeing Company, Rocketdyne Propulsion and Power.)

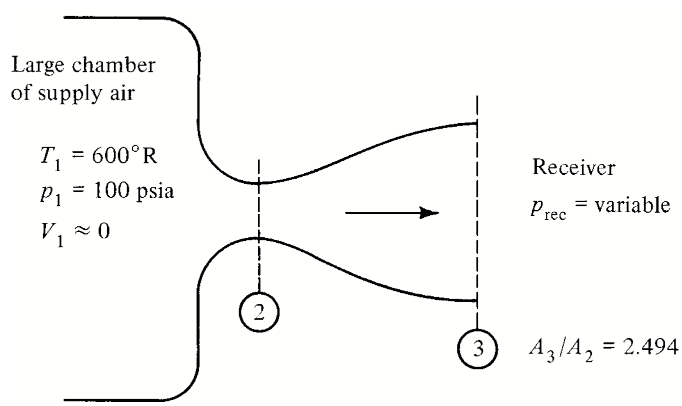

Figure 5.10 Converging–diverging nozzle.

To discover the conditions that exist at the exit (under design operation), we seek the ratio A_3/A_3^*:

$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2}\frac{A_2}{A_2^*}\frac{A_2^*}{A_3^*} = (2.494)(1)(1) = 2.494$$

Note that $A_2 = A_2^*$ since $M_2 = 1$, and $A_2^* = A_3^*$ by equation (5.29), as we are still assuming isentropic operation. We look for $A/A^* = 2.494$ in the *supersonic* section of the isentropic table and see that

$$M_3 = 2.44, \qquad \frac{p_3}{p_{t3}} = 0.0643, \qquad \text{and} \qquad \frac{T_3}{T_{t3}} = 0.4565$$

Thus

$$p_3 = \frac{p_3}{p_{t3}} \frac{p_{t3}}{p_{t1}} p_{t1} = (0.0643)(1)(100) = 6.43 \text{ psia}$$

and to operate the nozzle at this *design condition* the receiver pressure *must be* at 6.43 psia. The pressure variation through the nozzle for this case is shown as curve "a" in Figure 5.11. This mode is sometimes referred to as *third critical*. From the temperature ratio T_3/T_{t3} we can easily compute T_3, a_3, and V_3 by the procedure shown previously.

One can also find $A/A^* = 2.494$ in the subsonic section of the isentropic table. (Recall that these two answers come from the solution of a quadratic equation.) For this case

$$M_3 = 0.24, \qquad \frac{p_3}{p_{t3}} = 0.9607 \qquad \frac{T_3}{T_{t3}} = 0.9886$$

Thus

$$p_3 = \frac{p_3}{p_{t3}} \frac{p_{t3}}{p_{t1}} p_{t1} = (0.9607)(1)(100) = 96.07 \text{ psia}$$

and to operate at this condition the receiver pressure *must be* at 96.07 psia. With this receiver pressure the flow is subsonic from 1 to 2, sonic at 2, and *subsonic* again from

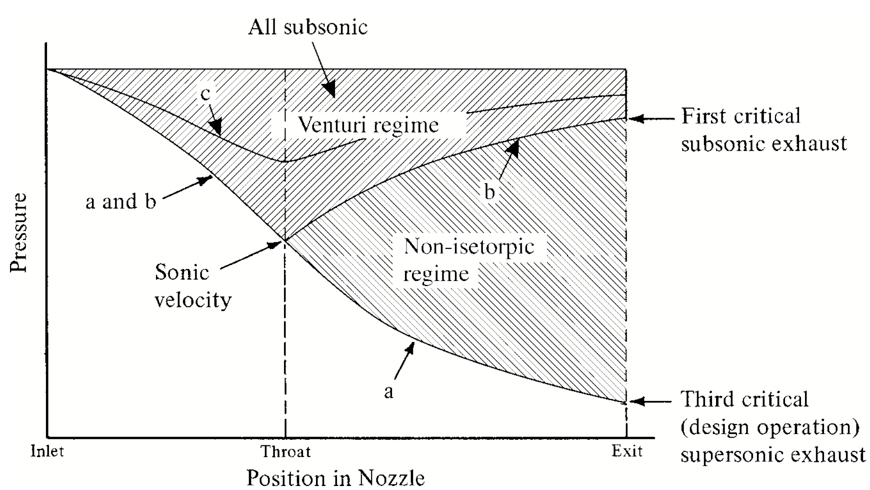

Figure 5.11 Pressure variation through converging–diverging nozzle.

2 to 3. The device is nowhere near its design condition and is really operating as a *venturi tube*; that is, the converging section is operating as a nozzle and the diverging section is operating as a diffuser. The pressure variation through the nozzle for this case is shown as curve "b" in Figure 5.11. This mode of operation is frequently called *first critical*.

Note that at both the first and third critical points, the flow variations are identical from the inlet to the throat. Once the receiver pressure has been lowered to 96.07 psia, Mach 1.0 exists in the throat and the device is said to be *choked*. *Further lowering of the receiver pressure will not change the flow rate.* Again, realize that it is not the pressure in the receiver by itself but rather the receiver pressure *relative* to the inlet pressure that determines the mode of operation.

Example 5.7 A converging–diverging nozzle with an area ratio of 3.0 exhausts into a receiver where the pressure is 1 bar. The nozzle is supplied by air at 22°C from a large chamber. At what pressure should the air in the chamber be for the nozzle to operate at its design condition (third critical point)? What will the outlet velocity be?

With reference to Figure 5.10, $A_3/A_2 = 3.0$:

$$\frac{A_3}{A_3^*} = \frac{A_3}{A_2}\frac{A_2}{A_2^*}\frac{A_2^*}{A_3^*} = (3.0)(1)(1) = 3.0$$

From the isentropic table:

$$M_3 = 2.64 \qquad \frac{p_3}{p_{t3}} = 0.0471 \qquad \frac{T_3}{T_{t3}} = 0.4177$$

$$p_1 = p_{t1} = \frac{p_{t1}}{p_{t3}}\frac{p_{t3}}{p_3} p_3 = (1)\left(\frac{1}{0.0471}\right)(1 \times 10^5) = 21.2 \times 10^5 \text{ N/m}^2$$

$$T_3 = \frac{T_3}{T_{t3}}\frac{T_{t3}}{T_{t1}} T_{t1} = (0.4177)(1)(22 + 273) = 123.2\text{K}$$

$$V_3 = M_3 a_3 = (2.64)\,[(1.4)(1)(287)(123.2)]^{1/2} = 587 \text{ m/s}$$

We have discussed only two specific operating conditions, and one might ask what happens at other receiver pressures. We can state that the first and third critical points represent the only operating conditions that satisfy the following criteria:

1. Mach 1 in the throat
2. Isentropic flow throughout the nozzle
3. Nozzle exit pressure equal to receiver pressure

With receiver pressures above the first critical, the nozzle operates as a venturi and we never reach sonic velocity in the throat. An example of this mode of operation is shown as curve "c" in Figure 5.11. The nozzle is no longer choked and the flow rate is less than the maximum. Conditions at the exit can be determined by the procedure shown previously for the converging-only nozzle. Then properties in the throat can be found if desired.

Operation between the first and third critical points is *not* isentropic. We shall learn later that under these conditions shocks will occur in either the diverging portion of the nozzle or after the exit. If the receiver pressure is below the third critical point, the nozzle operates *internally* as though it were at the design condition but expansion waves occur *outside* the nozzle. These operating modes will be discussed in detail as soon as the appropriate background has been developed.

5.8 NOZZLE PERFORMANCE

We have seen that the isentropic operating conditions are very easy to determine. Friction losses can then be taken into account by one of several methods. Direct information on the entropy change could be given, although this is usually not available. Sometimes equivalent information is provided in the form of the stagnation pressure ratio. Normally, however, nozzle performance is indicated by an *efficiency parameter*, which is defined as follows:

$$\boxed{\eta_n \equiv \frac{\text{actual change in kinetic energy}}{\text{ideal change in kinetic energy}}}$$

or

$$\eta_n \equiv \frac{\Delta \text{KE}_{\text{actual}}}{\Delta \text{KE}_{\text{ideal}}} \tag{5.45}$$

Since most nozzles involve negligible heat transfer (per unit mass of fluid flowing), we have from

$$h_{t1} + \not{q} = h_{t2} + \not{w}_s \tag{3.19}$$

$$h_{t1} = h_{t2} \tag{5.46}$$

Thus

$$h_1 + \frac{{V_1}^2}{2g_c} = h_2 + \frac{{V_2}^2}{2g_c} \tag{5.47a}$$

or

$$h_1 - h_2 = \frac{{V_2}^2 - {V_1}^2}{2g_c} \tag{5.47b}$$

Therefore, one normally sees the nozzle efficiency expressed as

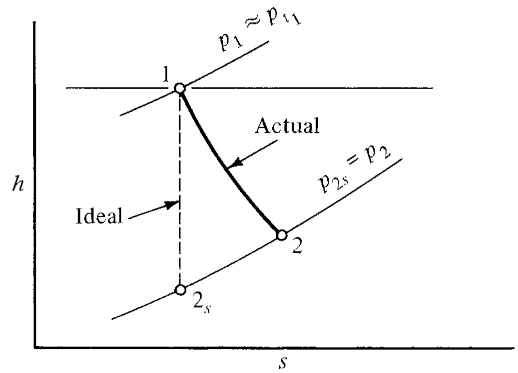

Figure 5.12 *h–s* diagram for a nozzle with losses.

$$\boxed{\eta_n = \frac{\Delta h_{\text{actual}}}{\Delta h_{\text{ideal}}}} \tag{5.48}$$

With reference to Figure 5.12, this becomes

$$\eta_n = \frac{h_1 - h_2}{h_1 - h_{2s}} \tag{5.49}$$

Since nozzle outlet velocities are quite large (relative to the velocity at the inlet), one can normally neglect the inlet velocity with little error. This is the case shown in Figure 5.12. Also note that the *ideal* process is assumed to take place down to the actual available receiver pressure. This definition of nozzle efficiency and its application appear quite reasonable since a nozzle is subjected to fixed (inlet and outlet) operating pressures and its purpose is to produce kinetic energy. The question is how well it does this, and η_n not only answers the question very quickly but permits a rapid determination of the actual outlet state.

Example 5.8 Air at 800°R and 80 psia feeds a converging-only nozzle having an efficiency of 96%. The receiver pressure is 50 psia. What is the actual nozzle outlet temperature?

Note that since $p_{\text{rec}}/p_{\text{inlet}} = 50/80 = 0.625 > 0.528$, the nozzle will not be choked, flow will be subsonic at the exit, and $p_2 = p_{\text{rec}}$ (see Figure 5.12).

$$\frac{p_{2s}}{p_{t2s}} = \frac{p_{2s}}{p_{t1}}\frac{p_{t1}}{p_{t2s}} = \left(\frac{50}{80}\right)(1) = 0.625$$

From table,

$$M_{2s} \approx 0.85 \quad \text{and} \quad \frac{T_{2s}}{T_{t2s}} = 0.8737$$

$$T_{2s} = \frac{T_{2s}}{T_{t2s}} \frac{T_{t2s}}{T_{t1}} T_{t1} = (0.8737)(1)(800) = 699°\text{R}$$

$$\eta_n = \frac{T_1 - T_2}{T_1 - T_{2s}} \qquad 0.96 = \frac{800 - T_2}{800 - 699}$$

$$T_2 = 703°\text{R}$$

Can you find the actual outlet velocity?

Another method of expressing nozzle performance is with a *velocity coefficient*, which is defined as

$$C_v \equiv \frac{\text{actual outlet velocity}}{\text{ideal outlet velocity}} \tag{5.50}$$

Sometimes a *discharge coefficient* is used and is defined as

$$C_d \equiv \frac{\text{actual mass flow rate}}{\text{ideal mass flow rate}} \tag{5.51}$$

5.9 DIFFUSER PERFORMANCE

Although the common use of nozzle efficiency makes this parameter well understood by all engineers, there is no single parameter that is universally employed for diffusers. Nearly a dozen criteria have been suggested to indicate diffuser performance (see p. 392, Vol. 1 of Ref 25). Two or three of these are the most popular, but unfortunately, even these are sometimes defined differently or called by different names. The following discussion refers to the h–s diagram shown in Figure 5.13.

Most of the propulsion industry uses the *total-pressure recovery factor* as a measure of diffuser performance. With reference to Figure 5.13, it is defined as

$$\boxed{\eta_r \equiv \frac{p_{t2}}{p_{t1}}} \tag{5.52}$$

This function is directly related to the area ratio A_1^*/A_2^* or the entropy change Δs_{1-2}, which we have previously shown to be equivalent loss indicators. As we shall see in Chapter 12, for propulsion applications this is usually referred to the free-stream conditions rather than the diffuser inlet.

For a definition of diffuser efficiency analogous to that of a nozzle, we recall that the function of a diffuser is to convert kinetic energy into pressure energy; thus it is logical to compare the ideal and actual processes between the same two enthalpy levels that represent the same kinetic energy change. Therefore, a suitable definition of *diffuser efficiency* is

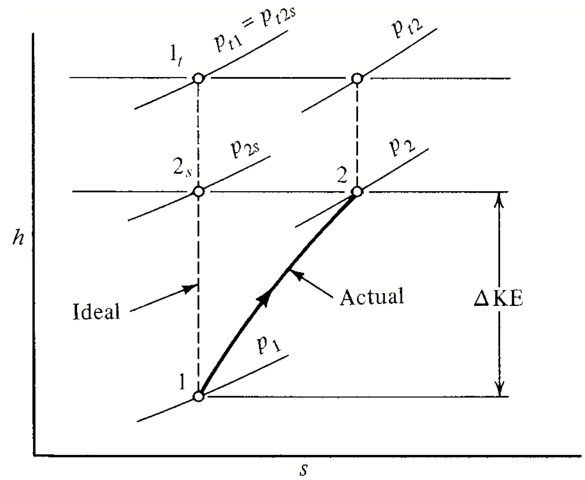

Figure 5.13 *h–s* diagram for a diffuser with losses.

$$\boxed{\eta_d \equiv \frac{\text{actual pressure rise}}{\text{ideal pressure rise}}} \tag{5.53}$$

or from Figure 5.13,

$$\eta_d \equiv \frac{p_2 - p_1}{p_{2s} - p_1} \tag{5.54}$$

You are again warned to be extremely cautious in accepting any performance figure for a diffuser without also obtaining a precise definition of what is meant by the criterion.

Example 5.9 A steady flow of air at 650°R and 30 psia enters a diffuser with a Mach number of 0.8. The total-pressure recovery factor $\eta_r = 0.95$. Determine the static pressure and temperature at the exit if $M = 0.15$ at that section.

With reference to Figure 5.13,

$$p_2 = \frac{p_2}{p_{t2}} \frac{p_{t2}}{p_{t1}} \frac{p_{t1}}{p_1} p_1 = (0.9844)(0.95)\left(\frac{1}{0.6560}\right)(30) = 42.8 \text{ psia}$$

$$T_2 = \frac{T_2}{T_{t2}} \frac{T_{t2}}{T_{t1}} \frac{T_{t1}}{T_1} T_1 = (0.9955)(1)\left(\frac{1}{0.8865}\right)(650) = 730°\text{R}$$

5.10 WHEN γ IS NOT EQUAL TO 1.4

In this section, as in the next few chapters, we present graphical information on one or more key parameter ratios as a function of the Mach number. This is done for various ratios of the specific heats ($\gamma = 1.13$, 1.4, and 1.67) to show the overall trends. Also, within a certain range of Mach numbers, the tabulations in Appendix G for air at normal temperature and pressure ($\gamma = 1.4$) which represent the middle of the range turn out to be satisfactory for other values of γ.

Figure 5.14 shows curves for p/p_t, T/T_t, and A/A^* in the interval $0.2 \leq M \leq 5$. Actually, compressible flow manifests itself in the range $M \geq 0.3$. Below this range we can treat flows as constant density (see Section 3.7 and Problem 4.3). Moreover, we have deliberately chosen to remain below the hypersonic range, which is generally regarded to be the region $M \geq 5$. So the interval chosen will be representative of many situations encountered in compressible flow. The curves in Figure 5.14 clearly show the important trends.

(a) As can be seen from Figure 5.14*a*, p/p_t is the least sensitive (of the three ratios plotted) to variations of γ. Below $M \approx 2.5$ the pressure ratio is well represented for any γ by the values tabulated in Appendix G.

(b) Figure 5.14*b* shows that T/T_t is more sensitive than the pressure ratio to variations of γ. But it shows relative insensitivity below $M \approx 0.8$ so that in this range the values tabulated in Appendix G could be used for any γ with little error.

(c) The same can be said about A/A^*, as shown in Figure 5.14*c*, which turns out to be relatively insensitive to variations in γ below $M \approx 1.5$.

In summary, the tables in Appendix G can be used for estimates (within ±5%) for almost any value of γ in the Mach number ranges identified above. Strictly speaking, these curves are representative only for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of changes are to be expected in other cases. Flows where γ variations are *not negligible within the flow* are treated in Chapter 11.

5.11 (OPTIONAL) BEYOND THE TABLES

Tables in gas dynamics are extremely useful but they have limitations, such as:

1. They do not show trends or the "big picture."
2. There is almost always the need for interpolation.
3. They display only one or at most a few values of γ.
4. They do not necessarily have the required accuracy.

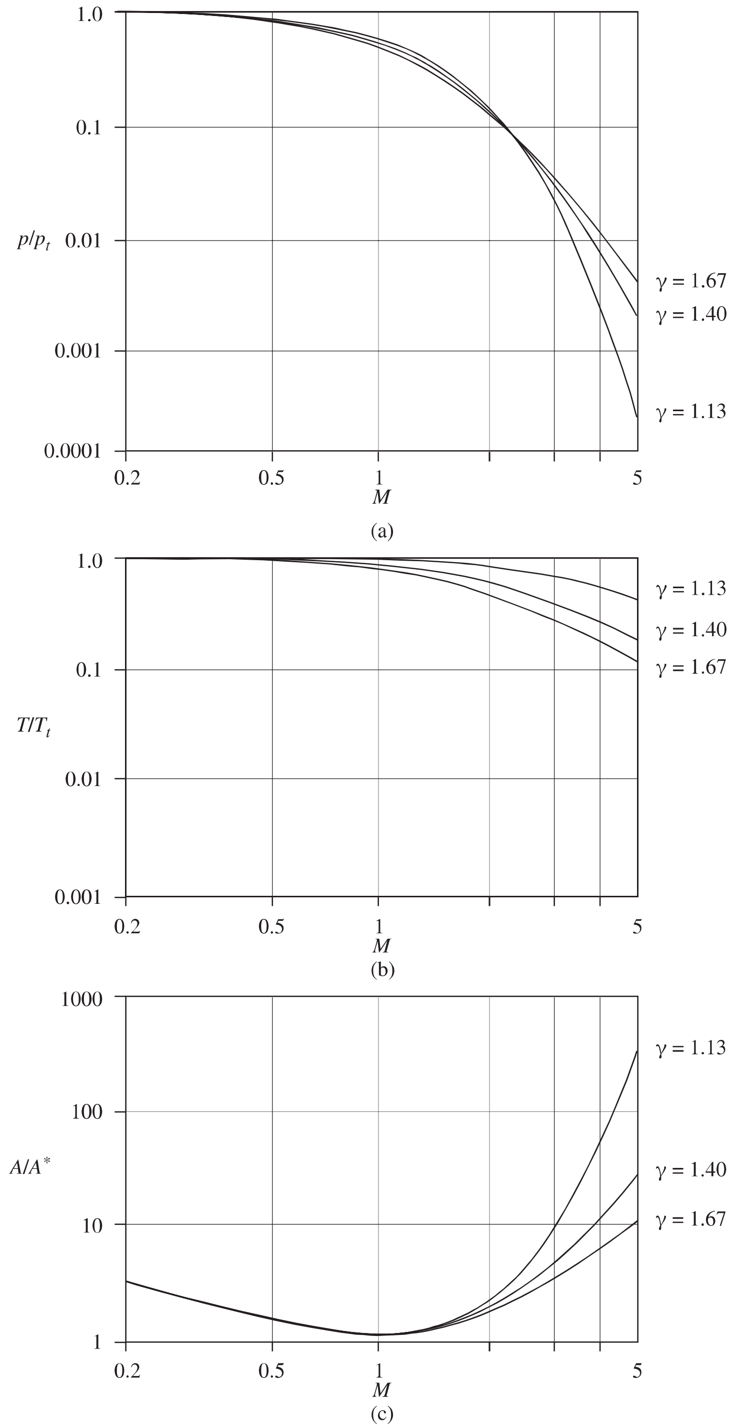

Figure 5.14 (a) Stagnation pressure ratio versus Mach number, (b) Stagnation temperature ratio versus Mach number, and (c) A/A^{*} area ratio versus Mach number for various values of γ.

Moreover, modern digital computers have made significant inroads in the working of problems, particularly when high-accuracy results and/or graphs are required. Simply put, the computer can be programmed to do the hard (and the easy) numerical calculations. In this book we have deliberately avoided integrating any gas dynamics software (some of which is commercially available) into the text material, preferring to present computer work as an adjunct to individual calculations. One reason is that we want you to spend your time learning about the wonderful world of gas dynamics and not on how to manage the programming. Another reason is that both computers and packaged software evolve too quickly, and therefore the attention that must be paid just to use any particular software is soon wasted.

Once you have mastered the basics, however, we feel that it is appropriate to discuss how things might be done with computers (and this could include handheld programmable calculators). In this book we discuss how the computer utility MAPLE can be of help in solving problems in gas dynamics. MAPLE is a powerful computer environment for doing symbolic, numerical, and graphical work. It is the product of Waterloo Maple, Inc., and the most recent version, MAPLE 7, was copyrighted in 2001. MAPLE is used routinely in many undergraduate engineering programs in the United States.

Other software packages are also popular in engineering schools. One in particular is MATLAB, which can do things equivalent to those handled by MAPLE. MATLAB's real forte is in manipulating linear equations and in constructing tables. But we have chosen MAPLE because it can manipulate equations symbolically and because of its superior graphics. In our view, this makes MAPLE somewhat more appropriate.

We will present some simple examples to show how MAPLE can be used. The experienced programmer can go much beyond these exercises. This section is optional because we want you to concentrate on the learning of gas dynamics and not spend extra time trying to demystify the computer approach. We focus on an example in Section 5.6, but the techniques must be understood to apply in general.

Example 5.10 In Example 5.6(a) the calculations can be done from the formulas or by using the tables for p_{t1} and T_{t1}. In part (b), however, direct calculation of M_2 given A_2/A_2^* is more difficult because it involves equation (5.37), which cannot be solved explicitly for M.

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+[(\gamma-1)/2]M^2}{(\gamma+1)/2}\right)^{(\gamma+1)/2(\gamma-1)} = f(M,\gamma) \qquad (5.37)$$

If we were given M_2, it would be simple to compute A_2/A_2^*.

But we are given A_2/A_2^* and we want to find M_2.

This is a problem where MAPLE can be useful because a built-in solver routine handles this type of problem easily.

First, we define some symbols: Let

$g \equiv \gamma$, a parameter (the ratio of the specific heats)

$X \equiv$ the independent variable (which in this case is M_2)

$Y \equiv$ the dependent variable (which in this case is A_2/A_2^*)

We need to introduce an index "*m*" to distinguish between subsonic and supersonic flow.

$$m \equiv \begin{cases} 1 & \text{for subsonic flow} \\ 10 & \text{for supersonic flow.} \end{cases}$$

Shown below is a copy of the precise MAPLE worksheet:

```
[ > g := 1.4:   Y := 1.1028:   m :=  10:
[ > fsolve(Y = (((1+(g-1)*(X^2)/2)/((g+1)/2)))^((g+1)/(2*(g-1)))/
    X, X, 1..m);
                                        1.377333281
```

which is the desired answer.

Here we discuss details of the MAPLE solution. If you are familiar with these, skip to the next paragraph. We must assume that the numerical value outputted is *X* because that is what we asked for in the executable statement with "fsolve( )," which terminates in a semicolon. Statements terminated in a colon are also executed but no return is asked for.

Example 5.11 We continue with this problem, as this is a good opportunity to show how MAPLE can help you avoid interpolation. If you are on the same worksheet, MAPLE remembers the values of g, Y, and X. We are now looking for the ratio of static to stagnation temperature, which is given the symbol Z. This ratio comes from equation (5.39):

$$\frac{T}{T_t} = \frac{1}{1 + [(\gamma - 1)/2]M^2} = f(M, \gamma) \tag{5.39}$$

Shown below are the precise inputs and program that you use in the computer.

```
[ > X := 1.3773:
[ > z := 1/(1 + (g-1)*(X^2)/2);
                                        Z := .7249575776
```

Now we can calculate the static temperature by the usual method.

$$T_2 = \frac{T_2}{T_{t2}} \frac{T_{t2}}{T_{t1}} T_{t1} = (0.725)(1)(1405) = 1019°\text{R}$$

The static pressure (p_2) can be found by a similar procedure.

5.12 SUMMARY

We analyzed a general varying-area configuration and found that properties vary in a radically different manner depending on whether the flow is subsonic or supersonic. The case of a perfect gas enabled the development of simple working equations for flow analysis. We then introduced the concept of a * reference state. The combination of the * and the stagnation reference states led to the development of the isentropic table, which greatly aids problem solution. Deviations from isentropic flow can be handled by appropriate loss factors or efficiency criteria.

A large number of useful equations were developed; however, most of these are of the type that need not be memorized. Equations (5.10), (5.11), and (5.13) were used for the general analysis of varying-area flow, and these are summarized in the middle of Section 5.3. The working equations that apply to a perfect gas are summarized at the end of Section 5.4 and are (4.28), (5.21), (5.23), (5.25), (5.27), and (5.28). Equations used as a basis for the isentropic table are numbered (5.37), (5.39), (5.40), (5.42), and (5.43) and are located in Section 5.6.

Those equations that are most frequently used are summarized below. You should be familiar with the conditions under which each may be used. Go back and review the equations listed in previous summaries, particularly those in Chapter 4.

1. *For steady one-dimensional flow of a perfect gas when $Q = W = 0$*

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

$$\frac{A_2^*}{A_1^*} = e^{\Delta s/R} \tag{5.29}$$

$$p_{t1} A_1^* = p_{t2} A_2^* \tag{5.35}$$

2. *Nozzle performance.*

Nozzle efficiency (between same pressures):

$$\eta_n \equiv \frac{\Delta \text{KE}_{\text{actual}}}{\Delta \text{KE}_{\text{ideal}}} = \frac{h_1 - h_2}{h_1 - h_{2s}} \tag{5.45), (5.49}$$

3. *Diffuser performance.*

Total-pressure recovery factor:

$$\eta_r \equiv \frac{p_{t2}}{p_{t1}} \tag{5.52}$$

or diffuser efficiency (between the same enthalpies):

$$\eta_d \equiv \frac{\text{actual pressure rise}}{\text{ideal pressure rise}} = \frac{p_2 - p_1}{p_{2s} - p_1} \tag{5.53), (5.54}$$

PROBLEMS

5.1. The following information is common to each of parts (a) and (b). Nitrogen flows through a diverging section with $A_1 = 1.5\ \text{ft}^2$ and $A_2 = 4.5\ \text{ft}^2$. You may assume steady, one- dimensional flow, $Q = W_s = 0$, negligible potential changes, and no losses.

(a) If $M_1 = 0.7$ and $p_1 = 70$ psia, find M_2 and p_2.

(b) If $M_1 = 1.7$ and $T_1 = 95°$F, find M_2 and T_2.

5.2. Air enters a converging section where $A_1 = 0.50$ m^2. At a downstream section $A_2 = 0.25$ m^2, $M_2 = 1.0$, and $\Delta s_{1-2} = 0$. It is known that $p_2 > p_1$. Find the initial Mach number (M_1) and the temperature ratio (T_2/T_1).

5.3. Oxygen flows into an insulated device with initial conditions as follows: $p_1 = 30$ psia, $T_1 = 750°$R, and $V_1 = 639$ ft/sec. The area changes from $A_1 = 6$ ft^2 to $A_2 = 5$ ft^2.

(a) Compute M_1, p_{t1}, and T_{t1}.

(b) Is this device a nozzle or diffuser?

(c) Determine M_2, p_2, and T_2 if there are no losses.

5.4. Air flows with $T_1 = 250$ K, $p_1 = 3$ bar abs., $p_{t1} = 3.4$ bar abs., and the cross-sectional area $A_1 = 0.40$ m^2. The flow is isentropic to a point where $A_2 = 0.30$ m^2. Determine the temperature at section 2.

5.5. The following information is known about the steady flow of air through an adiabatic system:

At section 1, $T_1 = 556°$R, $p_1, = 28.0$ psia

At section 2, $T_2 = 70°$F, $T_{t2} = 109°$F, $p_2, = 18$ psia

(a) Find M_2, V_2, and p_{t2}.

(b) Determine M_1, V_1, and p_{t1}.

(c) Compute the area ratio A_2/A_1.

(d) Sketch a physical diagram of the system along with a T–s diagram.

5.6. Assuming the flow of a perfect gas in an adiabatic, no-work system, show that sonic velocity corresponding to the stagnation conditions (a_t) is related to sonic velocity where the Mach number is unity (a^*) by the following equation:

$$\frac{a^*}{a_t} = \left(\frac{2}{\gamma + 1}\right)^{1/2}$$

5.7. Carbon monoxide flows through an adiabatic system. $M_1 = 4.0$ and $p_{t1} = 45$ psia. At a point downstream, $M_2 = 1.8$ and $p_2 = 7.0$ psia.

(a) Are there losses in this system? If so, compute Δs.

(b) Determine the ratio of A_2/A_1.

5.8. Two venturi meters are installed in a 30-cm-diameter duct that is insulated (Figure P5.8). The conditions are such that sonic flow exists at each throat (i.e., $M_1 = M_4 = 1.0$). Although each venturi is isentropic, the connecting duct has friction and hence losses exist between sections 2 and 3. $p_1 = 3$ bar abs. and $p_4 = 2.5$ bar abs. If the diameter at section 1 is 15 cm and the fluid is air:

(a) Compute Δs for the connecting duct.

(b) Find the diameter at section 4.

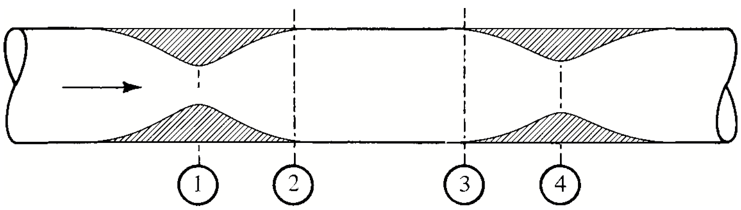

Figure P5.8

5.9. Starting with the flow rate as from equation (2.30), derive the following relation:

$$\frac{\dot{m}}{A} = M\left(1 + [(\gamma - 1)/2]M^2\right)^{-(\gamma+1)/2(\gamma-1)} \left(\frac{\gamma g_c}{R}\right)^{1/2} \frac{p_t}{\sqrt{T_t}}$$

5.10. A smooth 3-in.-diameter hole is punched into the side of a large chamber where oxygen is stored at 500°R and 150 psia. Assume frictionless flow.

(a) Compute the initial mass flow rate from the chamber if the surrounding pressure is 15.0 psia.

(b) What is the flow rate if the pressure of the surroundings is lowered to zero?

(c) What is the flow rate if the chamber pressure is raised to 300 psia?

5.11. Nitrogen is stored in a large chamber under conditions of 450 K and 1.5×10^5 N/m^2. The gas leaves the chamber through a convergent-only nozzle whose outlet area is 30 cm^2. The ambient room pressure is 1×10^5 N/m^2 and there are no losses.

(a) What is the velocity of the nitrogen at the nozzle exit?

(b) What is the mass flow rate?

(c) What is the maximum flow rate that could be obtained by lowering the ambient pressure?

5.12. A converging-only nozzle has an efficiency of 96%. Air enters with negligible velocity at a pressure of 150 psia and a temperature of 750°R. The receiver pressure is 100 psia. What are the actual outlet temperature, Mach number, and velocity?

5.13. A large chamber contains air at 80 psia and 600°R. The air enters a converging–diverging nozzle which has an area ratio (exit to throat) of 3.0.

(a) What pressure must exist in the receiver for the nozzle to operate at its first critical point?

(b) What should the receiver pressure be for third critical (design point) operation?

(c) If operating at its third critical point, what are the density and velocity of the air at the nozzle exit plane?

5.14. Air enters a convergent–divergent nozzle at 20 bar abs. and 40°C. At the end of the nozzle the pressure is 2.0 bar abs. Assume a frictionless adiabatic process. The throat area is 20 cm^2.

(a) What is the area at the nozzle exit?

(b) What is the mass flow rate in kg/s?

5.15. A converging–diverging nozzle is designed to operate with an exit Mach number of $M = 2.25$. It is fed by a large chamber of oxygen at 15.0 psia and 600°R and exhausts into the room at 14.7 psia. Assuming the losses to be negligible, compute the velocity in the nozzle throat.

5.16. A converging–diverging nozzle (Figure P5.16) discharges air into a receiver where the static pressure is 15 psia. A 1-ft^2 duct feeds the nozzle with air at 100 psia, 800°R, and a velocity such that the Mach number $M_1 = 0.3$. The exit area is such that the pressure at the nozzle exit exactly matches the receiver pressure. Assume steady, one-dimensional flow, perfect gas, and so on. The nozzle is adiabatic and there are no losses.

(a) Calculate the flow rate.

(b) Determine the throat area.

(c) Calculate the exit area.

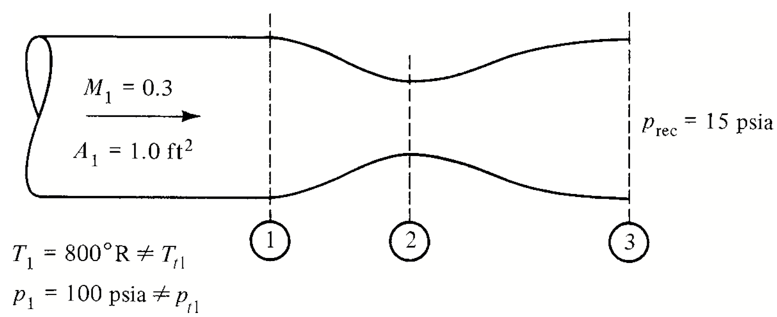

Figure P5.16

5.17. Ten kilograms per second of air is flowing in an adiabatic system. At one section the pressure is 2.0×10^5 N/m^2, the temperature is 650°C, and the area is 50 cm^2. At a downstream section $M_2 = 1.2$.

(a) Sketch the general shape of the system.

(b) Find A_2 if the flow is frictionless.

(c) Find A_2 if there is an entropy change between these two sections of 42 J/kg-K.

5.18. Carbon monoxide is expanded adiabatically from 100 psia, 540°F and negligible velocity through a converging–diverging nozzle to a pressure of 20 psia.

(a) What is the ideal exit Mach number?

(b) If the actual exit Mach number is found to be $M = 1.6$, what is the nozzle efficiency?

(c) What is the entropy change for the flow?

(d) Draw a T–s diagram showing the ideal and actual processes. Indicate pertinent temperatures, pressures, etc.

5.19. Air enters a converging–diverging nozzle with $T_1 = 22$°C, $p_1 = 10$ bar abs., and $V_1 \approx 0$. The exit Mach number is 2.0, the exit area is 0.25 m^2, and the nozzle efficiency is 0.95.

(a) What are the actual exit values of T, p, and p_t?

(b) What is the ideal exit Mach number?

(c) Assume that all the losses occur in the diverging portion of the nozzle and compute the throat area.

(d) What is the mass flow rate?

5.20. A diffuser receives air at 500°R, 18 psia, and a velocity of 750 ft/sec. The diffuser has an efficiency of 90% [as defined by equation (5.54)] and discharges the air with a velocity of 150 ft/sec.

(a) What is the pressure of the discharge air?

(b) What is the total-pressure recovery factor as given by equation (5.52)?

(c) Determine the area ratio of the diffuser.

5.21. Consider the steady, one-dimensional flow of a perfect gas through a horizontal system with no shaft work. No frictional losses are involved, but area changes and heat transfer effects provide a flow at constant temperature.

(a) Start with the pressure-energy equation and develop

$$\frac{p_2}{p_1} = e^{(\gamma/2)(M_1^2 - M_2^2)}$$

$$\frac{p_{t2}}{p_{t1}} = e^{(\gamma/2)(M_1^2 - M_2^2)} \left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2} \right)^{\gamma/(\gamma-1)}$$

(b) From the continuity equation show that

$$\frac{A_1}{A_2} = \frac{M_2}{M_1} e^{(\gamma/2)(M_1^2 - M_2^2)}$$

(c) By letting M_1 be any Mach number and $M_2 = 1.0$, write the expression for A/A^*. Show that the section of minimum area occurs at $M = 1/\sqrt{\gamma}$.

5.22. Consider the steady, one-dimensional flow of a perfect gas through a horizontal system with no heat transfer or shaft work. Friction effects are present, but area changes cause the flow to be at a constant Mach number.

(a) Recall the arguments of Section 4.6 and determine what other properties remain constant in this flow.

(b) Apply the concepts of continuity and momentum [equation (3.63)] to show that

$$D_2 - D_1 = \frac{f M^2 \gamma}{4}(x_2 - x_1)$$

You may assume a circular duct and a constant friction factor.

5.23. Assume that a supersonic nozzle operating isentropically delivers air at an exit Mach number of 2.8. The entrance conditions are 180 psia, 1000°R, and near-zero Mach number.

(a) Find the area ratio A_3/A_2 and the mass flow rate per unit throat area.

(b) What are the receiver pressure and temperature?

(c) If the entire diverging portion of the nozzle were suddenly to detach, what would the Mach number and $\dot{m}/A$ be at the new outlet?

5.24. Write a computer program and construct a table of isentropic flow parameters for $\gamma \neq 1.4$. (Useful values might be $\gamma = 1.2$, 1.3, or 1.67.) Use the following headings: M, p/p_t, T/T_t, ρ/ρ_t, A/A^*, and pA/p_tA^*. (*Hint:* Use MATLAB).

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

5.1. Define the * reference condition.

5.2. In adiabatic, no-work flow, the losses can be expressed by three different parameters. List these parameters and show how they are related to one another.

5.3. In the T–s diagram (Figure CT5.3), point 1 represents a stagnation condition. Proceeding isentropically from 1, the flow reaches a Mach number of unity at 1*. Point 2 represents another stagnation condition in the same flow system. Assuming that the fluid is a perfect gas, locate the corresponding isentropic 2* and prove that $T_2{}^*$ is either greater than, equal to, or less than $T_1{}^*$.

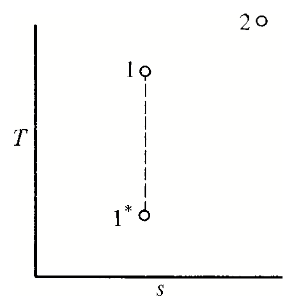

Figure CT5.3

5.4. A supersonic nozzle is fed by a large chamber and produces Mach 3.0 at the exit (Figure CT5.4). Sketch curves (to no particular scale) that show how properties vary through the nozzle as the Mach number increases from zero to 3.0.

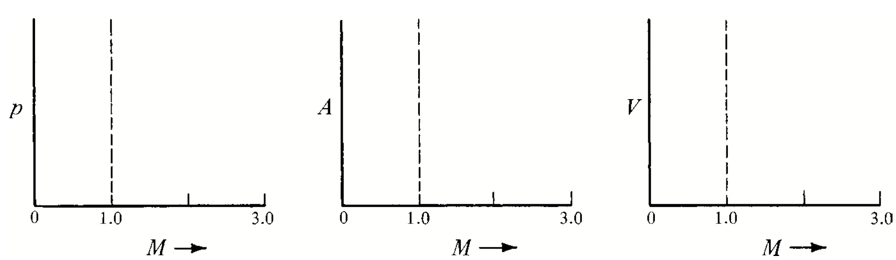

Figure CT5.4

5.5. Give a suitable definition for nozzle efficiency in terms of enthalpies. Sketch an h–s diagram to identify your state points.

5.6. Air flows steadily with no losses through a converging–diverging nozzle with an area ratio of 1.50. Conditions in the supply chamber are $T = 500°\text{R}$ and $p = 150$ psia.

(a) To choke the flow, to what pressure must the receiver be lowered?

(b) If the nozzle is choked, determine the density and velocity at the throat.

(c) If the receiver is at the pressure determined in part (a) and the diverging portion of the nozzle is removed, what will the exit Mach number be?

5.7. For steady, one-dimensional flow of a perfect gas in an adiabatic, no-work system, derive the working relation between the temperatures at two locations:

$$\frac{T_2}{T_1} = f(M_1, M_2, \gamma)$$

5.8. Work problem 5.20.

Chapter 6

Standing Normal Shocks

6.1 INTRODUCTION

Up to this point we have considered only continuous flows, flow systems in which state changes occur continuously and thus whose processes can easily be identified and plotted. Recall from Section 4.3 that *infinitesimal* pressure disturbances are called sound waves and these travel at a characteristic velocity that is determined by the medium and its thermodynamic state. In Chapters 6 and 7 we turn our attention to some *finite* pressure disturbances which are frequently encountered. Although incorporating large changes in fluid properties, the thickness of these disturbances is extremely small. Typical thicknesses are on the order of a few mean free molecular paths and thus they appear as *discontinuities* in the flow and are called *shock waves*.

Due to the complex interactions involved, analysis of the changes within a shock wave is beyond the scope of this book. Thus we deal only with the properties that exist on each side of the discontinuity. We first consider a *standing normal shock*, a stationary wave front that is perpendicular to the direction of flow. We will discover that this phenomenon is found only when supersonic flow exists and that it is basically a form of compression process. We apply the basic concepts of gas dynamics to analyze a shock wave in an arbitrary fluid and then develop working equations for a perfect gas. This procedure leads naturally to the compilation of tabular information which greatly simplifies problem solution. The chapter closes with a discussion of shocks found in the diverging portion of supersonic nozzles.

6.2 OBJECTIVES

After completing this chapter successfully, you should be able to:

1. List the assumptions used to analyze a standing normal shock.
2. Given the continuity, energy, and momentum equations for steady one-dimensional flow, utilize control volume analysis to derive the relations between properties on each side of a standing normal shock for an arbitrary fluid.

3. (*Optional*) Starting with the basic shock equations for an arbitrary fluid, derive the working equations for a perfect gas relating property ratios on each side of a standing normal shock as a function of Mach number (M) and specific heat ratio (γ).
4. (*Optional*) Given the working equations for a perfect gas, show that a unique relationship must exist between the Mach numbers before and after a standing normal shock.
5. (*Optional*) Explain how a normal-shock table may be developed that gives property ratios across the shock in terms of only the Mach number before the shock.
6. Sketch a normal-shock *process* on a T–s diagram, indicating as many pertinent features as possible, such as static and total pressures, static and total temperatures, and velocities. Indicate each of the preceding before and after the shock.
7. Explain why an *expansion shock* cannot exist.
8. Describe the second critical mode of nozzle operation. Given the area ratio of a converging–diverging nozzle, determine the operating pressure ratio that causes operation at the second critical point.
9. Describe how a converging–diverging nozzle operates between first and second critical points.
10. Demonstrate the ability to solve typical standing normal-shock problems by use of tables and equations.

6.3 SHOCK ANALYSIS—GENERAL FLUID

Figure 6.1 shows a standing normal shock in a section of varying area. We first establish a control volume that includes the shock region and an infinitesimal amount of fluid on each side of the shock. In this manner we deal only with the changes that occur across the shock. It is important to recognize that since the shock wave is so thin (about 10^{-6} m), a control volume chosen in the manner described above is extremely thin in the x-direction. This permits the following simplifications to be made without introducing error in the analysis:

1. The area on both sides of the shock may be considered to be the same.
2. There is negligible surface in contact with the wall, and thus frictional effects may be omitted.

We begin by applying the basic concepts of continuity, energy, and momentum under the following assumptions:

Steady one-dimensional flow	
Adiabatic	$\delta q = 0$ or $ds_e = 0$
No shaft work	$\delta w_s = 0$

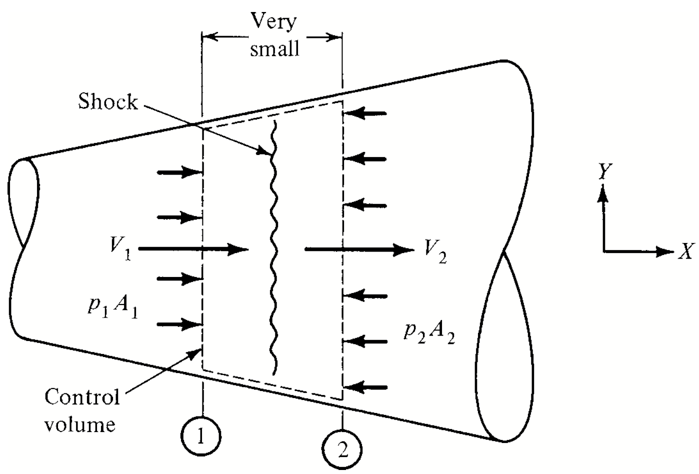

Figure 6.1 Control volume for shock analysis.

Neglect potential	$dz = 0$
Constant area	$A_1 = A_2$
Neglect wall shear	

Continuity

$$\dot{m} = \rho A V \tag{2.30}$$

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \tag{6.1}$$

But since the area is constant,

$$\boxed{\rho_1 V_1 = \rho_2 V_2} \tag{6.2}$$

Energy

We start with

$$h_{t1} + q = h_{t2} + w_s \tag{3.19}$$

For adiabatic and no work, this becomes

$$h_{t1} = h_{t2} \tag{6.3}$$

or

$$\boxed{h_1 + \frac{V_1^2}{2g_c} = h_2 + \frac{V_2^2}{2g_c}} \tag{6.4}$$

Momentum

The x-component of the momentum equation for steady one-dimensional flow is

$$\sum F_x = \frac{\dot{m}}{g_c}\left(V_{\text{out}_x} - V_{\text{in}_x}\right) \tag{3.46}$$

which when applied to Figure 6.1 becomes

$$\sum F_x = \frac{\dot{m}}{g_c}\left(V_{2x} - V_{1x}\right) \tag{6.5}$$

From Figure 6.1 we can also see that the force summation is

$$\sum F_x = p_1 A_1 - p_2 A_2 = (p_1 - p_2)A \tag{6.6}$$

Thus the momentum equation in the direction of flow becomes

$$(p_1 - p_2)A = \frac{\dot{m}}{g_c}(V_2 - V_1) = \frac{\rho A V}{g_c}(V_2 - V_1) \tag{6.7}$$

With $\dot{m}$ written as $\rho A V$, we can cancel the area from both sides. Now the ρV remaining can be written as either $\rho_1 V_1$ or $\rho_2 V_2$ [see equation (6.2)] and equation (6.7) becomes

$$p_1 - p_2 = \frac{\rho_2 V_2^2 - \rho_1 V_1^2}{g_c} \tag{6.8}$$

or

$$\boxed{p_1 + \frac{\rho_1 V_1^2}{g_c} = p_2 + \frac{\rho_2 V_2^2}{g_c}} \tag{6.9}$$

For the general case of an arbitrary fluid, we have arrived at *three* governing equations: (6.2), (6.4), and (6.9). A typical problem would be: Knowing the fluid and the conditions before the shock, predict the conditions that would exist after the shock. The unknown parameters are then *four* in number (ρ_2, p_2, h_2, V_2), which requires additional information for a problem solution. The missing information is supplied in the form of property relations for the fluid involved. For the general fluid (not a perfect gas), this leads to iterative-type solutions, but with modern digital computers these can be handled quite easily.

6.4 WORKING EQUATIONS FOR PERFECT GASES

In Section 6.3 we have seen that a typical normal-shock problem has four unknowns, which can be found through the use of the three governing equations (from continuity, energy, and momentum concepts) plus additional information on property relations. For the case of a perfect gas, this additional information is supplied in the form of an equation of state and the assumption of constant specific heats. We now proceed to develop working equations in terms of Mach numbers and the specific heat ratio.

Continuity

We start with the continuity equation developed in Section 6.3:

$$\rho_1 V_1 = \rho_2 V_2 \tag{6.2}$$

Substitute for the density from the perfect gas equation of state:

$$p = \rho R T \tag{1.13}$$

and for the velocity from equations (4.10) and (4.11):

$$V = Ma = M\sqrt{\gamma g_c R T} \tag{6.10}$$

Show that the continuity equation can now be written as

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \tag{6.11}$$

Energy

From Section 6.3 we have

$$h_{t1} = h_{t2} \tag{6.3}$$

But since we are now restricted to a perfect gas for which enthalpy is a function of temperature *only*, we can say that

$$T_{t1} = T_{t2} \tag{6.12}$$

Recall from Chapter 4 that for a perfect gas with constant specific heats,

$$T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right) \tag{4.18}$$

Hence the energy equation across a standing normal shock can be written as

$$T_1\left(1 + \frac{\gamma - 1}{2}{M_1}^2\right) = T_2\left(1 + \frac{\gamma - 1}{2}{M_2}^2\right) \tag{6.13}$$

Momentum

The momentum equation in the direction of flow was seen to be

$$p_1 + \frac{\rho_1 {V_1}^2}{g_c} = p_2 + \frac{\rho_2 {V_2}^2}{g_c} \tag{6.9}$$

Substitutions are made for the density from the equation of state (1.13) and for the velocity from equation (6.10):

$$p_1 + \left(\frac{p_1}{RT_1}\right)\left(\frac{{M_1}^2 \gamma g_c R T_1}{g_c}\right) = p_2 + \left(\frac{p_2}{RT_2}\right)\left(\frac{{M_2}^2 \gamma g_c R T_2}{g_c}\right) \tag{6.14}$$

and the momentum equation becomes

$$p_1\left(1 + \gamma M_1^2\right) = p_2\left(1 + \gamma M_2^2\right) \tag{6.15}$$

The governing equations for a standing normal shock have now been simplified for a perfect gas and for convenience are summarized below.

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} \tag{6.11}$$

$$T_1\left(1 + \frac{\gamma - 1}{2}{M_1}^2\right) = T_2\left(1 + \frac{\gamma - 1}{2}{M_2}^2\right) \tag{6.13}$$

$$p_1\left(1 + \gamma {M_1}^2\right) = p_2\left(1 + \gamma {M_2}^2\right) \tag{6.15}$$

There are seven variables involved in these equations:

$$\gamma,\ p_1,\ M_1,\ T_1,\ p_2,\ M_2,\ T_2$$

Once the gas is identified, γ is known, and a given state preceding the shock fixes p_1, M_1, and T_1. Thus equations (6.11), (6.13), and (6.15) are sufficient to solve for the unknowns after the shock: p_2, M_2, and T_2.

Rather than struggle through the details of the solution for every shock problem that we encounter, let's solve it once and for all right now. We proceed to combine the equations above and derive an expression for M_2 in terms of the information given. First, we rewrite equation (6.11) as

$$\frac{p_1 M_1}{p_2 M_2} = \sqrt{\frac{T_1}{T_2}} \tag{6.16}$$

and equation (6.13) as

$$\sqrt{\frac{T_1}{T_2}} = \left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{1/2} \tag{6.17}$$

and equation (6.15) as

$$\frac{p_1}{p_2} = \frac{1 + \gamma M_2^2}{1 + \gamma M_1^2} \tag{6.18}$$

We then substitute equations (6.17) and (6.18) into equation (6.16), which yields

$$\left(\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2}\right)\frac{M_1}{M_2} = \left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{1/2} \tag{6.19}$$

At this point notice that M_2 is a function of only M_1 and γ. A trivial solution of this is seen to be $M_1 = M_2$, which represents the degenerate case of no shock. To solve the nontrivial case, we square equation (6.19), cross-multiply, and arrange the result as a quadratic in M_2^2:

$$A\left(M_2^2\right)^2 + BM_2^2 + C = 0 \tag{6.20}$$

where A, B, and C are functions of M_1 and γ. Only if you have considerable motivation should you attempt to carry out the tedious algebra (or to utilize a computer utility, see Section 6.9) required to show that the solution of this quadratic is

$$\boxed{M_2^2 = \frac{M_1^2 + 2/(\gamma - 1)}{[2\gamma/(\gamma - 1)]M_1^2 - 1}} \tag{6.21}$$

For our typical shock problem the Mach number after the shock is computed with the aid of equation (6.21), and then T_2 and p_2 can easily be found from equations (6.13) and (6.15). To complete the picture, the total pressures p_{t1} and p_{t2} can be computed in the usual manner. It turns out that since M_1 is supersonic, M_2

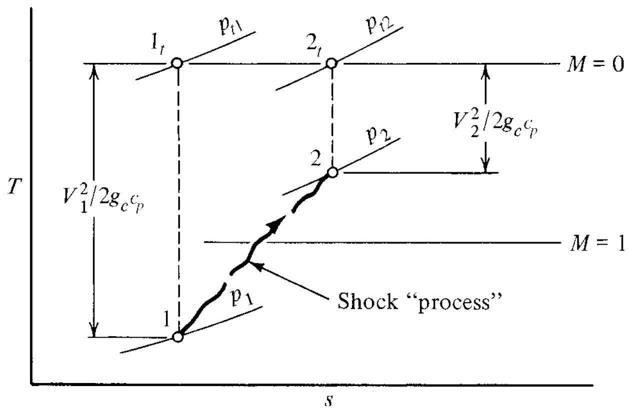

Figure 6.2 T–s diagram for typical normal shock.

will always be subsonic and a typical problem is shown on the T–s diagram in Figure 6.2.

The end points 1 and 2 (before and after the shock) are well-defined states, but the changes that occur within the shock do not follow an equilibrium process in the usual thermodynamic sense. For this reason the shock *process* is usually shown by a dashed or wiggly line. Note that when points 1 and 2 are located on the T–s diagram, it can immediately be seen that an entropy change is involved in the shock process. This is discussed in greater detail in the next section.

Example 6.1 Helium is flowing at a Mach number of 1.80 and enters a normal shock. Determine the pressure ratio across the shock.

We use equation (6.21) to find the Mach number after the shock and (6.15) to obtain the pressure ratio.

$$M_2^2 = \frac{M_1^2 + 2/(\gamma - 1)}{[2\gamma/(\gamma - 1)]M_1^2 - 1} = \frac{(1.8)^2 + 2/(1.67 - 1)}{[(2 \times 1.67)/(1.67 - 1)](1.8)^2 - 1} = 0.411$$

$$M_2 = 0.641$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} = \frac{1 + (1.67)(1.8)^2}{1 + (1.67)(0.411)} = 3.80$$

6.5 NORMAL-SHOCK TABLE

We have found that for any given fluid with a specific set of conditions entering a normal shock there is one and only one set of conditions that can result after the shock. An iterative solution results for a fluid that cannot be treated as a perfect gas, whereas the case of the perfect gas produces an explicit solution. The latter case opens the door to further simplifications since equation (6.21) yields the exit Mach number

M_2 for any given inlet Mach number M_1 and we can now eliminate M_2 from all previous equations.

For example, equation (6.13) can be solved for the temperature ratio

$$\frac{T_2}{T_1} = \frac{1 + [(\gamma - 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_2^2} \tag{6.22}$$

If we now eliminate M_2 by the use of equation (6.21), the result will be

$$\frac{T_2}{T_1} = \frac{\{1 + [(\gamma - 1)/2]M_1^2\}\{[2\gamma/(\gamma - 1)]M_1^2 - 1\}}{[(\gamma + 1)^2/2(\gamma - 1)]M_1^2} \tag{6.23}$$

Similarly, equation (6.15) can be solved for the pressure ratio

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{6.24}$$

and elimination of M_2 through the use of equation (6.21) will produce

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1} \tag{6.25}$$

If you are very persistent (and in need of algebraic exercise or want to do it with a computer), you might carry out the development of equations (6.23) and (6.25). Also, these can be combined to form the density ratio

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \tag{6.26}$$

Other interesting ratios can be developed, each as a function of only M_1 and γ. For example, since

$$p_t = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma - 1)} \tag{4.21}$$

we may write

$$\frac{p_{t2}}{p_{t1}} = \frac{p_2}{p_1}\left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{\gamma/(\gamma - 1)} \tag{6.27}$$

The ratio p_2/p_1 can be eliminated by equation (6.25) with the following result:

$$\frac{p_{t2}}{p_{t1}} = \left(\frac{[(\gamma + 1)/2]M_1^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{\gamma/(\gamma - 1)} \left[\frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1}\right]^{1/(1 - \gamma)} \tag{6.28}$$

Equation (6.28) is extremely important since the stagnation pressure ratio is related to the entropy change through equation (4.28):

$$\frac{p_{t2}}{p_{t1}} = e^{-\Delta s/R} \tag{4.28}$$

In fact, we could combine equations (4.28) and (6.28) to obtain an explicit relation for Δs as a function of M_1 and γ.

Note that for a given fluid (γ known), the equations (6.23), (6.25), (6.26), and (6.28) express property ratios as a function of the entering Mach number only. This suggests that we could easily construct a table giving values of M_2, T_2/T_1, p_2/p_1, ρ_2/ρ_1, p_{t2}/p_{t1}, and so on, versus M_1 for a particular γ. Such a table of normal-shock parameters is given in Appendix H. This table greatly aids problem solution, as the following example shows.

Example 6.2 Fluid is air and can be treated as a perfect gas. If the conditions before the shock are: $M_1 = 2.0$, $p_1 = 20$ psia, and $T_1 = 500°\text{R}$; determine the conditions after the shock and the entropy change across the shock.

First we compute p_{t1} with the aid of the isentropic table.

$$p_{t1} = \frac{p_{t1}}{p_1} p_1 = \left(\frac{1}{0.1278}\right)(20) = 156.5 \text{ psia}$$

Now from the normal-shock table opposite $M_1 = 2.0$, we find

$$M_2 = 0.57735 \qquad \frac{p_2}{p_1} = 4.5000 \qquad \frac{T_2}{T_1} = 1.6875 \qquad \frac{p_{t2}}{p_{t1}} = 0.72087$$

Thus

$$p_2 = \frac{p_2}{p_1} p_1 = (4.5)(20) = 90 \text{ psia}$$

$$T_2 = \frac{T_2}{T_1} T_1 = (1.6875)(500) = 844°\text{R}$$

$$p_{t2} = \frac{p_{t2}}{p_{t1}} p_{t1} = (0.72087)(156.5) = 112.8 \text{ psia}$$

Or p_{t2} can be computed with the aid of the isentropic table:

$$p_{t2} = \frac{p_{t2}}{p_2} p_2 = \left(\frac{1}{0.7978}\right)(90) = 112.8 \text{ psia}$$

To compute the entropy change, we use equation (4.28):

$$\frac{p_{t2}}{p_{t1}} = 0.72087 = e^{-\Delta s/R}$$

$$\frac{\Delta s}{R} = 0.3273$$

$$\Delta s = \frac{(0.3273)(53.3)}{778} = 0.0224 \text{ Btu/lbm-°R}$$

It is interesting to note that as far as the governing equations are concerned, the problem in Example 6.2 could be completely reversed. The fundamental relations of continuity (6.11), energy (6.13), and momentum (6.15) would be satisfied completely if we changed the problem to $M_1 = 0.577$, $p_1 = 90$ psia, $T_1 = 844$°R, with the resulting $M_2 = 2.0$, $p_2 = 20$ psia, and $T_2 = 500$°R (which would represent an *expansion shock*). However, in the latter case the entropy change would be *negative*, which clearly violates the second law of thermodynamics for an adiabatic no-work system.

Example 6.2 and the accompanying discussion clearly show that the shock phenomenon is a one-way process (i.e., irreversible). It is always a compression shock, and for a normal shock the flow is always supersonic before the shock and subsonic after the shock. One can note from the table that as M_1 increases, the pressure, temperature, and density ratios increase, indicating a stronger shock (or compression). One can also note that as M_1 increases, p_{t2}/p_{t1} decreases, which means that the entropy change increases. Thus *as the strength of the shock increases, the losses also increase*.

Example 6.3 Air has a temperature and pressure of 300 K and 2 bar abs., respectively. It is flowing with a velocity of 868 m/s and enters a normal shock. Determine the density before and after the shock.

$$\rho_1 = \frac{p_1}{RT_1} = \frac{2 \times 10^5}{(287)(300)} = 2.32 \text{ kg/m}^3$$

$$a_1 = (\gamma g_c R T_1)^{1/2} = [(1.4)(1)(287)(300)]^{1/2} = 347 \text{ m/s}$$

$$M_1 = \frac{V_1}{a_1} = \frac{868}{347} = 2.50$$

From the shock table we obtain

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2} = (7.125)\left(\frac{1}{2.1375}\right) = 3.333$$

$$\rho_2 = 3.3333\rho_1 = (3.3333)(2.32) = 7.73 \text{ kg/m}^3$$

Example 6.4 Oxygen enters the converging section shown in Figure E6.4 and a normal shock occurs at the exit. The entering Mach number is 2.8 and the area ratio $A_1/A_2 = 1.7$. Compute the overall static temperature ratio T_3/T_1. Neglect all frictional losses.

$$\frac{A_2}{A_2^*} = \frac{A_2}{A_1}\frac{A_1}{A_1^*}\frac{A_1^*}{A_2^*} = \left(\frac{1}{1.7}\right)(3.5001)(1) = 2.06$$

Thus $M_2 \approx 2.23$, and from the shock table we get

$$M_3 = 0.5431 \quad \text{and} \quad \frac{T_3}{T_2} = 1.8835$$

$$\frac{T_3}{T_1} = \frac{T_3}{T_2}\frac{T_2}{T_{t2}}\frac{T_{t2}}{T_{t1}}\frac{T_{t1}}{T_1} = (1.8835)(0.5014)(1)\frac{1}{0.3894} = 2.43$$

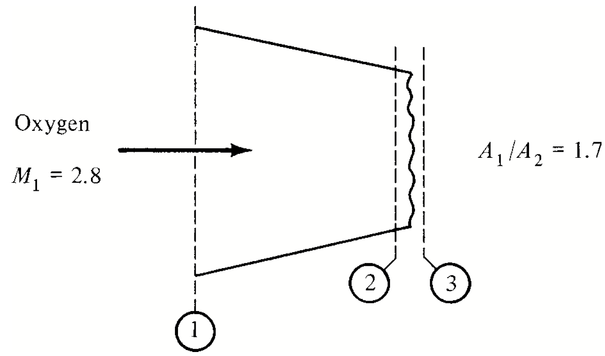

Figure E6.4

We can also develop a relation for the velocity change across a standing normal shock for use in Chapter 7. Starting with the basic continuity equation

$$\rho_1 V_1 = \rho_2 V_2 \tag{6.2}$$

we introduce the density relation from (6.26):

$$\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma - 1)M_1^2 + 2}{(\gamma + 1)M_1^2} \tag{6.29}$$

and subtract 1 from each side:

$$\frac{V_2 - V_1}{V_1} = \frac{(\gamma - 1)M_1^2 + 2 - (\gamma + 1)M_1^2}{(\gamma + 1)M_1^2} \tag{6.30}$$

$$\frac{V_2 - V_1}{M_1 a_1} = \frac{2\left(1 - M_1^2\right)}{(\gamma + 1)M_1^2} \tag{6.31}$$

or

$$\boxed{\frac{V_1 - V_2}{a_1} = \left(\frac{2}{\gamma + 1}\right)\left(\frac{M_1^2 - 1}{M_1}\right)} \tag{6.32}$$

This is another parameter that is a function of M_1 and γ and thus may be added to our shock table. Its usefulness for solving certain types of problems will become apparent in Chapter 7.

6.6 SHOCKS IN NOZZLES

In Section 5.7 we discussed the isentropic operations of a converging–diverging nozzle. Remember that this type of nozzle is physically distinguished by its *area ratio*, the ratio of the exit area to the throat area. Furthermore, its flow conditions are determined by the *operating pressure ratio*, the ratio of the receiver pressure to the inlet stagnation pressure. We identified two significant critical pressure ratios. For any pressure ratio above the first critical point, the nozzle is not choked and has subsonic flow throughout (typical venturi operation). The first critical point represents flow that is subsonic in both the convergent and divergent sections but is choked with a Mach number of 1.0 in the throat. The third critical point represents operation at the design condition with subsonic flow in the converging section and supersonic flow in the entire diverging section. It is also choked with Mach 1.0 in the throat. The first and third critical points are the only operating points that have (1) isentropic flow throughout, (2) a Mach number of 1 at the throat, and (3) exit pressure equal to receiver pressure.

Remember that with subsonic flow at the exit, the exit pressure *must* equal the receiver pressure. Imposing a pressure ratio slightly below that of the first critical point presents a problem in that there is no way that *isentropic* flow can meet the boundary condition of pressure equilibrium at the exit. However, there is nothing to prevent a *nonisentropic* flow adjustment from occurring within the nozzle. This internal adjustment takes the form of a standing normal shock, which we now know involves an entropy change.

As the pressure ratio is lowered below the first critical point, a normal shock forms just downstream of the throat. The remainder of the *nozzle* is now acting as a diffuser since after the shock the flow is subsonic and the area is increasing. The shock will locate itself in a position such that the pressure changes that occur ahead of the shock, across the shock, and downstream of the shock will produce a pressure that *exactly matches the outlet pressure*. In other words, *the operating pressure ratio determines the location and strength of the shock*. An example of this mode of operation is shown in Figure 6.3. As the pressure ratio is lowered further, the shock continues to move toward the exit. When the shock is located at the exit plane, this condition is referred to as the *second critical point*.

We have ignored boundary layer effects that are always present due to fluid viscosity. These effects sometimes cause what are known as *lambda shocks*. It is important for you to understand that real flows are often much more complicated than the idealizations that we are describing.

If the operating pressure ratio is between the second and third critical points, a compression takes place *outside* the nozzle. This is called *overexpansion* (i.e., the flow has been expanded too far within the nozzle). If the receiver pressure is below the third critical point, an expansion takes place *outside* the nozzle. This condition is called *underexpansion*. We investigate these conditions in Chapters 7 and 8 after the appropriate background has been covered.

For the present we proceed to investigate the operational regime between the first and second critical points. Let us work with the same nozzle and inlet conditions that

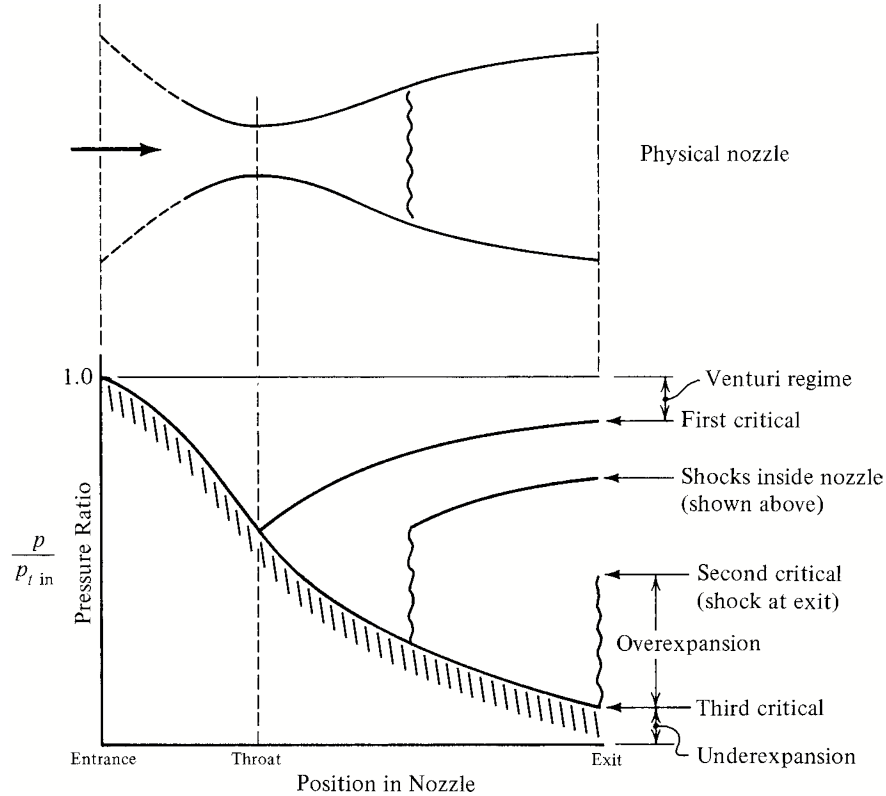

Figure 6.3 Operating modes for DeLaval nozzle.

we used in Section 5.7. The nozzle has an area ratio of 2.494 and is fed by air at 100 psia and 600°R from a large tank. Thus the inlet conditions are essentially stagnation. For these fixed inlet conditions we previously found that a receiver pressure of 96.07 psia (an *operating pressure ratio* of 0.9607) identifies the first critical point and a receiver pressure of 6.426 psia (an *operating pressure ratio* of 0.06426) exists at the third critical point.

What receiver pressure do we need to operate at the second critical point? Figure 6.4 shows such a condition and you should recognize that the entire nozzle up to the shock is operating at its design or third critical condition.

From the isentropic table at $A/A^* = 2.494$, we have

$$M_3 = 2.44 \quad \text{and} \quad \frac{p_3}{p_{t3}} = 0.06426$$

From the normal-shock table for $M_3 = 2.44$, we have

$$M_4 = 0.5189 \quad \text{and} \quad \frac{p_4}{p_3} = 6.7792$$

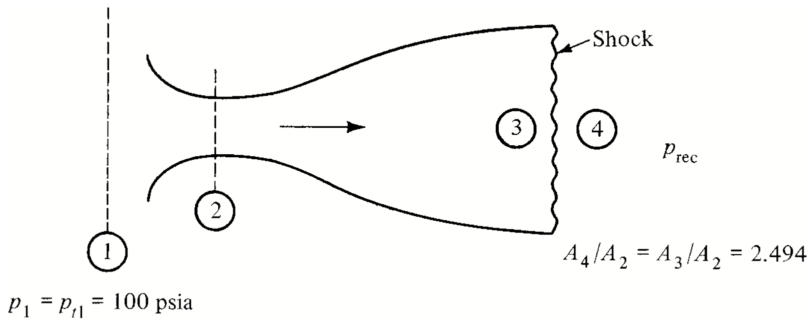

Figure 6.4 Operation at second critical.

and the operating pressure ratio will be

$$\frac{p_{rec}}{p_{t1}} = \frac{p_4}{p_{t1}} = \frac{p_4}{p_3}\frac{p_3}{p_{t3}}\frac{p_{t3}}{p_{t1}} = (6.7792)(0.06426)(1) = 0.436$$

or for $p_1 = p_{t1} = 100$ psia,

$$p_4 = p_{rec} = 43.6 \text{ psia}$$

Thus for our converging–diverging nozzle with an area ratio of 2.494, any operating pressure ratio between 0.9607 and 0.436 will cause a normal shock to be located someplace in the diverging portion of the nozzle.

Suppose that we are given an operating pressure ratio of 0.60. The logical question to ask is: Where is the shock? This situation is shown in Figure 6.5. We must take advantage of the only two available pieces of information and from these construct a solution. We know that

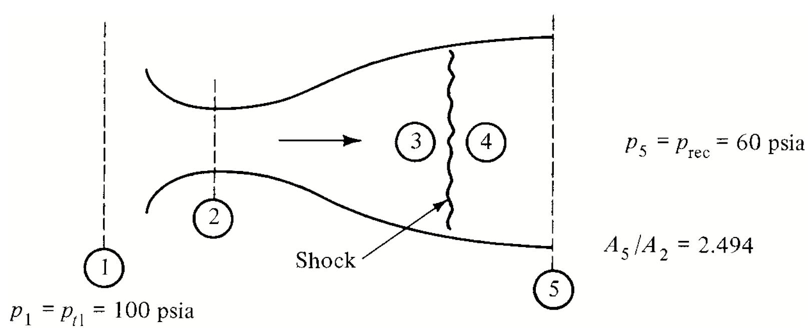

Figure 6.5 DeLaval nozzle with normal shock in diverging section.

$$\frac{A_5}{A_2} = 2.494 \quad \text{and} \quad \frac{p_5}{p_{t1}} = 0.60$$

We may also assume that all losses occur across the shock and we know that $M_2 = 1.0$. It might also be helpful to visualize the flow on a T–s diagram, and this is shown in Figure 6.6. Since there are no losses up to the shock, we know that

$$A_2 = A_1^*$$

Thus

$$\frac{A_5}{A_2}\frac{p_5}{p_{t1}} = \frac{A_5}{A_1^*}\frac{p_5}{p_{t1}} \tag{6.33}$$

We also know from equation (5.35) that for the case of adiabatic no-work flow of a perfect gas,

$$A_1^* p_{t1} = A_5^* p_{t5} \tag{6.34}$$

Thus

$$\frac{A_5 p_5}{A_1^* p_{t1}} = \frac{A_5 p_5}{A_5^* p_{t5}}$$

In summary:

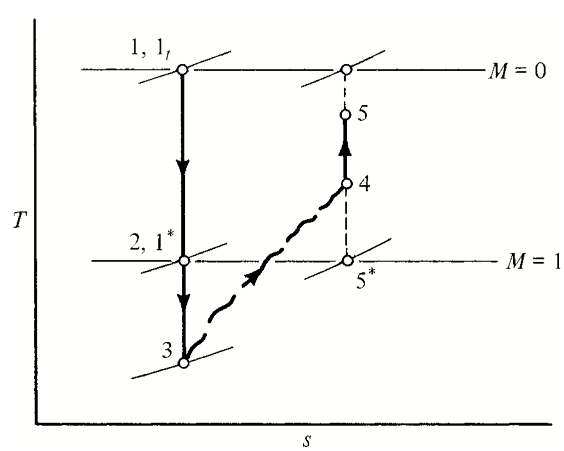

Figure 6.6 T–s diagram for DeLaval nozzle with normal shock. (For physical picture see Figure 6.5.)

$$\frac{A_5}{A_2}\frac{p_5}{p_{t1}} = \frac{A_5}{A_1^*}\frac{p_5}{p_{t1}} = \frac{A_5 p_5}{A_5^* p_{t5}} \qquad (6.35)$$

known

$$(2.494)(0.6) = 1.4964$$

Note that we have manipulated the known information into an expression with all similar station subscripts. In Section 5.6 we showed with equation (5.43) that the ratio $Ap/A^* p_t$ is a simple function of M and γ and thus is listed in the isentropic table. A check in the table shows that the exit Mach number is $M_5 \approx 0.38$.

To locate the shock, seek the ratio

$$\frac{p_{t5}}{p_{t1}} = \frac{p_{t5}}{p_5}\frac{p_5}{p_{t1}} = \left(\frac{1}{0.9052}\right)(0.6) = 0.664$$

Given (p_5/p_{t1})

From isentropic table at $M = 0.38$ (p_{t5}/p_5)

and since all the loss is assumed to take place across the shock, we have

$$p_{t5} = p_{t4} \quad \text{and} \quad p_{t1} = p_{t3}$$

Thus

$$\frac{p_{t4}}{p_{t3}} = \frac{p_{t5}}{p_{t1}} = 0.664$$

Knowing the total pressure ratio across the shock, we can determine from the normal-shock table that $M_3 \approx 2.12$, and then from the isentropic table we note that this Mach number will occur at an area ratio of about $A_3/A_3^* = A_3/A_2 = 1.869$. More accurate answers could be obtained by interpolating within the tables.

We see that if we are given a physical converging–diverging nozzle (area ratio is known) and an operating pressure ratio between the first and second critical points, it is a simple matter to determine the position and strength of the normal shock in the diverging section.

Example 6.5 A converging–diverging nozzle has an area ratio of 3.50. At off-design conditions, the exit Mach number is observed to be 0.3. What operating pressure ratio would cause this situation?

Using the section numbering system of Figure 6.5, for $M_3 = 0.3$, we have

$$\frac{p_5 A_5}{p_{t5} A_5^*} = 1.9119$$

$$\frac{p_5}{p_{t1}} = \frac{p_5 A_5}{p_{t5} A_5^*}\left(\frac{p_{t5} A_5^*}{p_{t1} A_1^*}\right)\frac{A_1^*}{A_2}\frac{A_2}{A_5} = (1.9119)(1)(1)\left(\frac{1}{3.50}\right) = 0.546$$

Could you now find the shock location and Mach number?

Example 6.6 Air enters a converging–diverging nozzle that has an overall area ratio of 1.76. A normal shock occurs at a section where the area is 1.19 times that of the throat. Neglect all friction losses and find the operating pressure ratio. Again, we use the numbering system shown in Figure 6.5.

From the isentropic table at $A_3/A_2 = 1.19$, $M_3 = 1.52$.

From the shock table, $M_4 = 0.6941$ and $p_{t4}/p_{t3} = 0.9233$. Then

$$\frac{A_5}{A_5^*} = \frac{A_5}{A_2}\frac{A_2}{A_4}\frac{A_4}{A_4^*}\frac{A_4^*}{A_5^*} = (1.76)\left(\frac{1}{1.19}\right)(1.0988)(1) = 1.625$$

Thus $M_5 \approx 0.389$.

$$\frac{p_5}{p_{t1}} = \frac{p_5}{p_{t5}}\frac{p_{t5}}{p_{t4}}\frac{p_{t4}}{p_{t3}}\frac{p_{t3}}{p_{t1}} = (0.9007)(1)(0.9233)(1) = 0.832$$

6.7 SUPERSONIC WIND TUNNEL OPERATION

To provide a test section with supersonic flow requires a converging–diverging nozzle. To operate economically, the nozzle–test-section combination must be followed by a diffusing section which also must be converging–diverging. This configuration presents some interesting problems in flow analysis. Starting up such a wind tunnel is another example of nozzle operation at pressure ratios above the second critical point. Figure 6.7 shows a typical tunnel in its most *unfavorable* operating condition, which occurs at startup. A brief analysis of the situation follows.

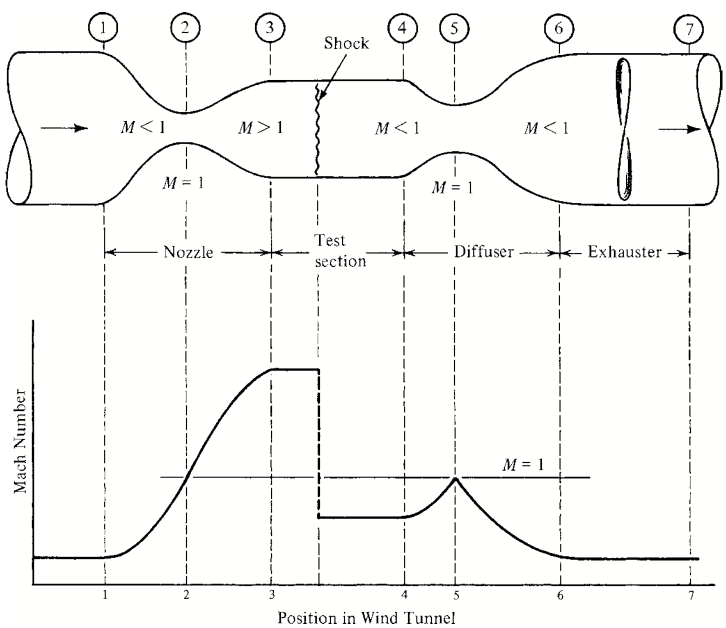

Figure 6.7 Supersonic tunnel at startup (with associated Mach number variation).

As the exhauster is started, this reduces the pressure and produces flow through the tunnel. At first the flow is subsonic throughout, but at increased power settings the exhauster reduces pressures still further and causes increased flow rates until the nozzle throat (section 2) becomes choked. At this point the nozzle is operating at its first critical condition. As power is increased further, a normal shock is formed just downstream of the throat, and if the tunnel pressure is decreased continuously, the shock will move down the diverging portion of the nozzle and pass rapidly through the test section and into the diffuser. Figure 6.8 shows this general running condition, which is called the *most favorable condition*.

We return to Figure 6.7, which shows the shock located in the test section. The variation of Mach number throughout the flow system is also shown for this case. This is called the *most unfavorable condition* because the shock occurs at the highest possible Mach number and thus the losses are greatest. We might also point out that the diffuser throat (section 5) must be sized for this condition. Let us see how this is done.

Recall the relation $p_t A^* =$ constant. Thus

$$p_{t2}A_2^* = p_{t5}A_5^*$$

But since Mach 1 exists at both sections 2 and 5 (during startup),

$$A_2 = A_2^* \quad \text{and} \quad A_5 = A_5^*$$

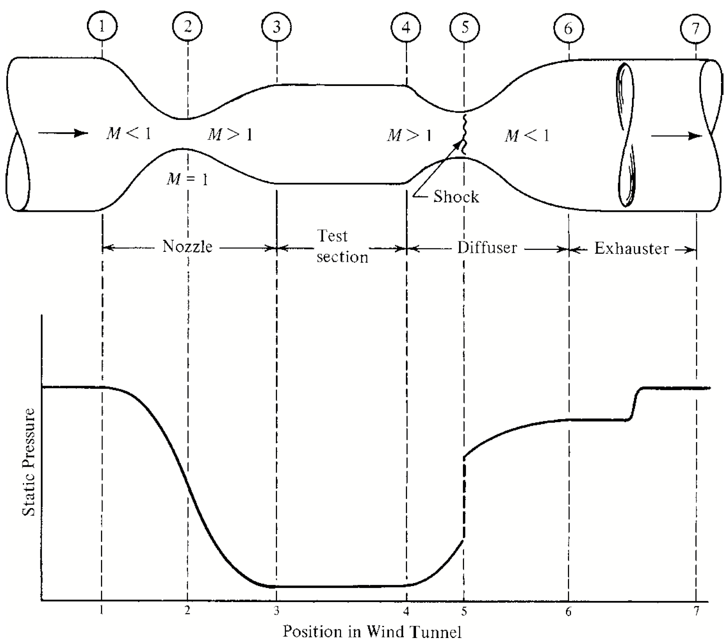

Figure 6.8 Supersonic tunnel in running condition (with associated pressure variation).

Hence

$$p_{t2}A_2 = p_{t5}A_5 \tag{6.36}$$

Due to the shock losses (and other friction losses), we know that $p_{t5} < p_{t2}$, and therefore A_5 must be greater than A_2. Knowing the test-section-design Mach number fixes the shock strength in this unfavorable condition and A_5 is easily determined from equation (6.36). Keep in mind that this represents a *minimum* area for the diffuser throat. If it is made any smaller than this, the tunnel could never be started (i.e., we could never get the shock into and through the test section). In fact, if A_5 is made too small, the flow will choke first in this throat and never get a chance to reach sonic conditions in section 2.

Once the shock has passed into the diffuser throat, knowing that $A_5 > A_2$ we realize that the tunnel can never run with sonic velocity at section 5. Thus, to operate as a diffuser, there must be a shock at this point, as shown in Figure 6.8. We have also shown the pressure variation through the tunnel for this running condition.

To keep the losses during running at a minimum, the shock in the diffuser should occur at the lowest possible Mach number, which means a small throat. However, we have seen that it is necessary to have a large diffuser throat in order to start the tunnel. A solution to this dilemma would be to construct a diffuser with a variable-area throat. After startup, A_5 could be decreased, with a corresponding decrease in shock strength and operating power. However, the power required for any installation must always be computed on the basis of the unfavorable startup condition.

Although the supersonic wind tunnel is used primarily for aeronautically oriented work, its operation serves to solidify many of the important concepts of variable-area flow, normal shocks, and their associated flow losses. Equally important is the fact that it begins to focus our attention on some practical design applications.

6.8 WHEN γ IS NOT EQUAL TO 1.4

As indicated in Chapter 5, we discuss the effects that changes from $\gamma = 1.4$ bring about. Figures 6.9 and 6.10 show curves for T_2/T_1 and p_2/p_1 versus Mach number in the interval $1 \leq M \leq 5$ entering the shock. This is done for various ratios of the specific heats ($\gamma = 1.13$, 1.4, and 1.67).

1. Figure 6.9 depicts T_2/T_1 across a normal-shock wave. As can be seen in the figure, the temperature ratio is very sensitive to γ.
2. On the other hand, as shown in Figure 6.10, the pressure ratio across the normal shock is relatively less sensitive to γ. Below $M \approx 1.5$ the pressure ratio tabulated in Appendix H could be used with little error for any γ.

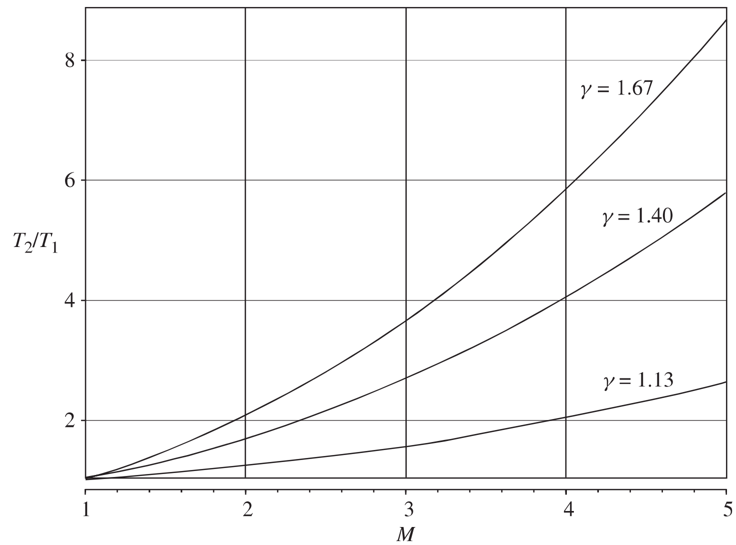

Figure 6.9 Temperature ratio across a normal shock versus Mach number for various values of γ.

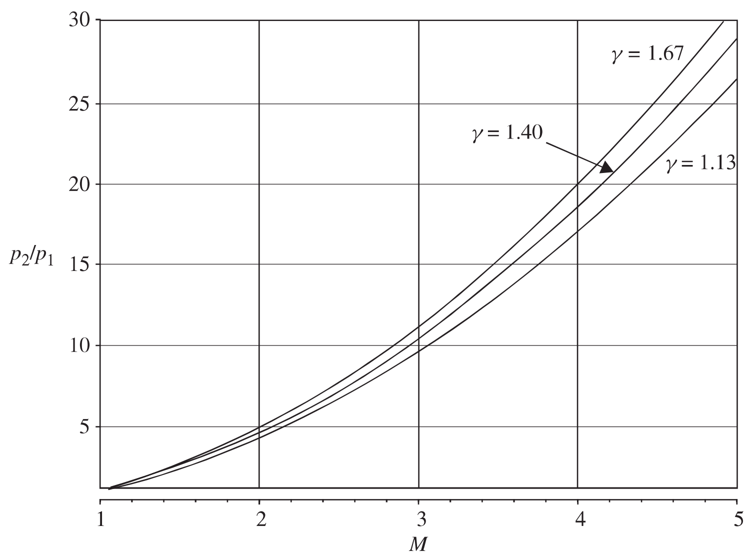

Figure 6.10 Pressure ratio across a normal shock versus Mach number for various values of γ.

Strictly speaking, these curves are representative only for cases where γ variations are *negligible within the flow*. However, they offer hints as to what magnitude of changes are to be expected in other cases. Flows where γ-variations are *not negligible within the flow* are treated in Chapter 11.

6.9 (OPTIONAL) BEYOND THE TABLES

As illustrated in Chapter 5, one can eliminate a lot of interpolation and get accurate answers for any ratio of the specific heats γ and/or any Mach number by using a computer utility such as MAPLE. For instance, we can easily calculate the left-hand side of equations (6.21), (6.23), (6.25), (6.26), and (6.28) to a high degree of precision given M_1 and γ (or calculate any one of the three variables given the other two).

Example 6.7 Let's go back to Example 6.3, where the density ratio across the shock is desired. We can compute this from equation (6.26):

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{(\gamma - 1)M_1^2 + 2} \tag{6.26}$$

Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)

$X \equiv$ the independent variable (which in this case is M_1)

$Y \equiv$ the dependent variable (which in this case is ρ_2/ρ_1)

Listed below are the precise inputs and program that you use in the computer.

```
[> g := 1.4:  X := 2.5:
[> Y := ((g+1)*X^2)/((g-1)*X^2 + 2);
                                     Y := 3.333333333
```

which is the desired answer.

A rather unique capability of MAPLE is its ability to solve equations symbolically (in contrast to strictly numerically). This comes in handy when trying to reproduce proofs of somewhat complicated algebraic expressions.

Example 6.8 Suppose that we want to solve for M_2 in equation (6.19):

$$\left(\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2}\right)\frac{M_1}{M_2} = \left(\frac{1 + [(\gamma - 1)/2]M_2^2}{1 + [(\gamma - 1)/2]M_1^2}\right)^{1/2} \tag{6.19}$$

Let

$g \equiv \gamma$, a parameter (the ratio of specific heats)

$X \equiv$ the independent variable (which in this case is M_1^2)

$$Y \equiv \text{ the dependent variable (which in this case is } M_2{}^2)$$

Listed below are the precise inputs and program that you use in the computer.

```
> solve((((1 + g*Y)^2)/((1 + g*X)^2))*(X/Y) = (2 +
  (g - 1)*Y)/(2 + (g-1)*X), Y);
```

$$X, \frac{2 + Xg - X}{-g + 1 + 2Xg}$$

which are the desired answers.

Above are the two roots of Y (or $M_2{}^2$), because we are solving a quadratic. With some manipulation we can get the second or nontrivial root to look like equation (6.21). It is easy to check it by substituting in some numbers and comparing results with the normal-shock table.

The type of calculation shown above can be integrated into more sophisticated programs to handle most gas dynamic calculations.

6.10 SUMMARY

We examined stationary discontinuities of a type perpendicular to the flow. These are finite pressure disturbances and are called *standing normal shock waves*. If conditions are known ahead of a shock, a precise set of conditions must exist after the shock. Explicit solutions can be obtained for the case of a perfect gas and these lend themselves to tabulation for various specific heat ratios.

Shocks are found only in supersonic flow, and the flow is always subsonic after a normal shock. The shock wave is a type of compression process, although a rather inefficient one since relatively large losses are involved in the process. (What has been lost?) Shocks provide a means of flow adjustment to meet imposed pressure conditions in supersonic flow.

As in Chapter 5, most of the equations in this chapter need not be memorized. However, you should be completely familiar with the fundamental relations that apply to all fluids across a normal shock. These are equations (6.2), (6.4), and (6.9). Essentially, these say that the end points of a shock have three things in common:

1. The same mass flow per unit area
2. The same stagnation enthalpy
3. The same value of $p + \rho V^2/g_c$

The working equations that apply to perfect gases, equations (6.11), (6.13), and (6.15), are summarized in Section 6.4. In Section 6.5 we developed equation (6.32) and noted that it can be very useful in solving certain types of problems. You should also be familiar with the various ratios that have been tabulated in Appendix H. Just knowing what kind of information you have available is frequently very helpful in setting up a problem solution.

PROBLEMS

Unless otherwise indicated, you may assume that there is no friction in any of the following flow systems; thus the only losses are those generated by shocks.

6.1. A standing normal shock occurs in air that is flowing at a Mach number of 1.8.

(a) What are the pressure, temperature, and density ratios across the shock?

(b) Compute the entropy change for the air as it passes through the shock.

(c) Repeat part (b) for flows at $M = 2.8$ and 3.8.

6.2. The difference between the total and static pressure before a shock is 75 psi. What is the maximum static pressure that can exist at this point ahead of the shock? The gas is oxygen. (*Hint:* Start by finding the static and total pressures ahead of the shock for the limiting case of $M = 1.0$.)

6.3. In an arbitrary perfect gas, the Mach number before a shock is infinite.

(a) Determine a general expression for the Mach number after the shock. What is the value of this expression for $\gamma = 1.4$?

(b) Determine general expressions for the ratios p_2/p_1, T_2/T_1, ρ_2/ρ_1, and p_{t2}/p_{t1}. Do these agree with the values shown in Appendix H for $\gamma = 1.4$?

6.4. It is known that sonic velocity exists in each throat of the system shown in Figure P6.4. The entropy change for the air is 0.062 Btu/lbm-°R. Negligible friction exists in the duct. Determine the area ratios A_3/A_1 and A_2/A_1.

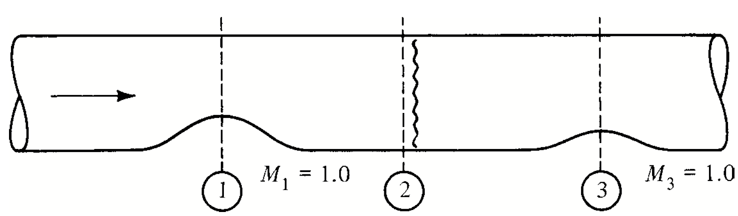

Figure P6.4

6.5. Air flows in the system shown in Figure P6.5. It is known that the Mach number after the shock is $M_3 = 0.52$. Considering p_1 and p_2, it is also known that one of these pressures is twice the other.

(a) Compute the Mach number at section 1.

(b) What is the area ratio A_1/A_2?

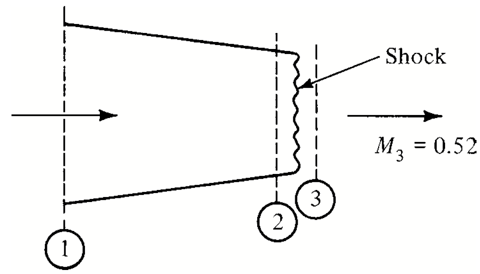

Figure P6.5

6.6. A shock stands at the inlet to the system shown in Figure P6.6. The free-stream Mach number is $M_1 = 2.90$, the fluid is nitrogen, $A_2 = 0.25\ \text{m}^2$, and $A_3 = 0.20\ \text{m}^2$. Find the outlet Mach number and the temperature ratio T_3/T_1.

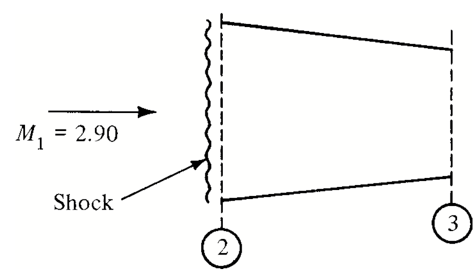

Figure P6.6

6.7. A converging–diverging nozzle is designed to produce a Mach number of 2.5 with air.

(a) What operating pressure ratio (p_{rec}/p_t inlet) will cause this nozzle to operate at the first, second, and third critical points?

(b) If the inlet stagnation pressure is 150 psia, what receiver pressures represent operation at these critical points?

(c) Suppose that the receiver pressure were fixed at 15 psia. What inlet pressures are necessary to cause operation at the critical points?

6.8. Air enters a convergent–divergent nozzle at $20 \times 10^5\ \text{N/m}^2$ and 40°C. The receiver pressure is $2 \times 10^5\ \text{N/m}^2$ and the nozzle throat area is $10\ \text{cm}^2$.

(a) What should the exit area be for the design conditions above (i.e., to operate at third critical?)

(b) With the nozzle area fixed at the value determined in part (a) and the inlet pressure held at $20 \times 10^5\ \text{N/m}^2$, what receiver pressure would cause a shock to stand at the exit?

(c) What receiver pressure would place the shock at the throat?

6.9. In Figure P6.9, $M_1 = 3.0$ and $A_1 = 2.0$ ft^2. If the fluid is carbon monoxide and the shock occurs at an area of 1.8 ft^2, what is the minimum area possible for section 4?

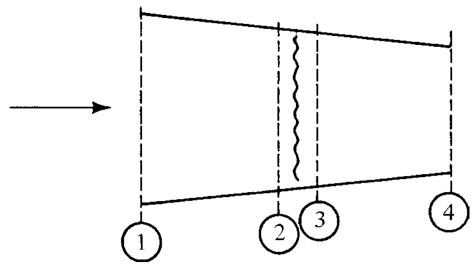

Figure P6.9

6.10. A converging–diverging nozzle has an area ratio of 7.8 but is not being operated at its design pressure ratio. Consequently, a normal shock is found in the diverging section at an area twice that of the throat. The fluid is oxygen.

(a) Find the Mach number at the exit and the operating pressure ratio.

(b) What is the entropy change through the nozzle if there is negligible friction?

6.11. The diverging section of a supersonic nozzle is formed from the frustrum of a cone. When operating at its third critical point with nitrogen, the exit Mach number is 2.6. Compute the operating pressure ratio that will locate a normal shock as shown in Figure P6.11.

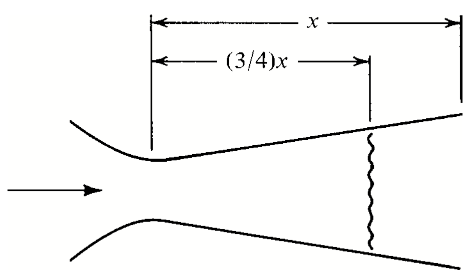

Figure P6.11

6.12. A converging–diverging nozzle receives air from a tank at 100 psia and 600°R. The pressure is 28.0 psia immediately preceding a plane shock that is located in the diverging section. The Mach number at the exit is 0.5 and the flow rate is 10 lbm/sec. Determine:

(a) The throat area.

(b) The area at which the shock is located.

(c) The outlet pressure required to operate the nozzle in the manner described above.

(d) The outlet area.

(e) The design Mach number.

6.13. Air enters a device with a Mach number of $M_1 = 2.0$ and leaves with $M_2 = 0.25$. The ratio of exit to inlet area is $A_2/A_1 = 3.0$.

(a) Find the static pressure ratio p_2/p_1.

(b) Determine the stagnation pressure ratio p_{t2}/p_{t1}.

6.14. Oxygen, with $p_t = 95.5$ psia, enters a diverging section of area 3.0 ft^2. At the outlet the area is 4.5 ft^2, the Mach number is 0.43, and the static pressure is 75.3 psia. Determine the possible values of Mach number that could exist at the inlet.

6.15. A converging–diverging nozzle has an area ratio of 3.0. The stagnation pressure at the inlet is 8.0 bar and the receiver pressure is 3.5 bar. Assume that $\gamma = 1.4$.

(a) Compute the critical operating pressure ratios for the nozzle and show that a shock is located within the diverging section.

(b) Compute the Mach number at the outlet.

(c) Compute the shock location (area) and the Mach number before the shock.

6.16. Nitrogen flows through a converging–diverging nozzle designed to operate at a Mach number of 3.0. If it is subjected to an operating pressure ratio of 0.5:

(a) Determine the Mach number at the exit.

(b) What is the entropy change in the nozzle?

(c) Compute the area ratio at the shock location.

(d) What value of the operating pressure ratio would be required to move the shock to the exit?

6.17. Consider a converging–diverging nozzle feeding air from a reservoir at p_1 and T_1. The exit area is $A_e = 4A_2$, where A_2 is the area at the throat. The back pressure p_{rec} is steadily reduced from an initial $p_{\text{rec}} = p_1$.

(a) Determine the receiver pressures (in terms of p_1) that would cause this nozzle to operate at first, second, and third critical points.

(b) Explain how the nozzle would be operating at the following back pressures:

(i) $p_{\text{rec}} = p_1$; **(ii)** $p_{\text{rec}} = 0.990p_1$; **(iii)** $p_{\text{rec}} = 0.53p_1$; **(iv)** $p_{\text{rec}} = 0.03p_1$.

6.18. Draw a detailed T–s diagram corresponding to the *supersonic tunnel startup* condition (Figure 6.7). Identify the various stations (i.e., 1, 2, 3, etc.) in your diagram. You may assume no heat transfer and no frictional losses in the system.

6.19. Consider the wind tunnel shown in Figures 6.7 and 6.8. Atmospheric air enters the system with a pressure and temperature of 14.7 psia and 80°F, respectively, and has negligible velocity at section 1. The test section has a cross-sectional area of 1 ft^2 and operates at a Mach number of 2.5. You may assume that the diffuser reduces the velocity to approximately zero and that final exhaust is to the atmosphere with negligible velocity. The system is fully insulated and there are negligible friction losses. Find:

(a) The throat area of the nozzle.

(b) The mass flow rate.

(c) The minimum possible throat area of the diffuser.

(d) The total pressure entering the exhauster at startup (Figure 6.7).

(e) The total pressure entering the exhauster when running (Figure 6.8).

(f) The hp value required for the exhauster (based on an isentropic compression).

CHECK TEST

You should be able to complete this test without reference to material in the chapter.

6.1. Given the continuity, energy, and momentum equations in a form suitable for steady one-dimensional flow, analyze a standing normal shock in an arbitrary fluid. Then simplify your results for the case of a perfect gas.

6.2. Fill in the following blanks with *increases*, *decreases*, or *remains constant*. Across a standing normal shock, the

(a) Temperature __________

(b) Stagnation pressure __________

(c) Velocity __________

(d) Density __________

6.3. Consider a converging–diverging nozzle with an area ratio of 3.0 and assume operation with a perfect gas ($\gamma = 1.4$). Determine the operating pressure ratios that would cause operation at the first, second, and third critical points.

6.4. Sketch a T–s diagram for a standing normal shock in a perfect gas. Indicate static and total pressures, static and total temperatures, and velocities (both before and after the shock).

6.5. Nitrogen flows in an insulated variable-area system with friction. The area ratio is $A_2/A_1 = 2.0$ and the static pressure ratio is $p_2/p_1 = 0.20$. The Mach number at section 2 is $M_2 = 3.0$.

(a) What is the Mach number at section 1?

(b) Is the gas flowing from 1 to 2 or from 2 to 1?

6.6. A large chamber contains air at 100 psia and 600°R. A converging–diverging nozzle with an area ratio of 2.50 is connected to the chamber and the receiver pressure is 60 psia.

(a) Determine the outlet Mach number and velocity.

(b) Find the Δs value across the shock.

(c) Draw a T–s diagram for the flow through the nozzle.